Mapping General Residue theorems to IR-equations for NMHV amplitudes in $\mathcal{N} = 4$ SYM

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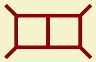
Amplitudes in $\mathcal{N} = 4$ SYM

-  all tree-level amplitudes known [Parke Taylor][Drummond Henn Korchemsky Sokatchev]
-  one-loop amplitudes known [Bern Dixon Kosower][Drummond Henn Korchemsky Sokatchev]
-  some results available, but no general (supersymmetric) form known [Bern Dixon Kosower Roiban Spradlin Volovich Vergu][Vergu]

- infrared behaviour well explored for one-loop amplitudes
- two-loop: no integral basis singled out, infrared behaviour for single amplitudes known.

Conventions:

- Kinematical invariants:
$$s_{i\,i+1\,\ldots\,i+m} = (p_i + \ldots + p_{i+m})^2$$
$$s_{ij} = \langle ij \rangle [ij] = 2k_i \cdot k_j$$
- Spinor helicity formalism:
$$p^{\alpha\dot{\alpha}} = p_\mu (\sigma^{\alpha\dot{\alpha}})^\mu,\ p_\mu p^\mu = \det(p^{\alpha\dot{\alpha}})$$
$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}},\ \langle ij \rangle = \lambda_i^\alpha \lambda_{j\alpha},\ [ij] = \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}}$$

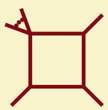
Infrared singularities at one loop

- consistency equations for infrared divergences

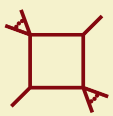
$$M^{1\text{-loop}}\Big|_{IR} = \sum_i C_i I^i\Big|_{IR} = -\frac{1}{\epsilon^2}\sum_{i=1}^{n}(-s_{i,i+1})^{-\epsilon} M^{\text{tree}}.$$

  in $D = 4 - \epsilon$.

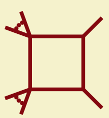
- IR-divergent part of the integral basis:



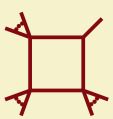
$$I^{1\text{m}}(p,q,r,P) = -\tfrac{1}{\epsilon^2}\left((-s)^{-\epsilon} + (-t)^{-\epsilon} - (-P^2)^{-\epsilon}\right)$$



$$I^{2\text{me}}(p,P,q,Q) = -\tfrac{1}{\epsilon^2}\left((-s)^{-\epsilon} + (-t)^{-\epsilon} - (-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}\right)$$



$$I^{2\text{mh}}(p,q,P,Q) = -\tfrac{1}{\epsilon^2}\left(\tfrac{1}{2}(-s)^{-\epsilon} + (-t)^{-\epsilon} - \tfrac{1}{2}(-P^2)^{-\epsilon} - \tfrac{1}{2}(-Q^2)^{-\epsilon}\right)$$



$$I^{3\text{m}}(p,P,R,Q) = -\tfrac{1}{\epsilon^2}\left(\tfrac{1}{2}(-s)^{-\epsilon} + \tfrac{1}{2}(-t)^{-\epsilon} - \tfrac{1}{2}(-P^2)^{-\epsilon} - \tfrac{1}{2}(-Q^2)^{-\epsilon}\right)\ ,$$

  where $s = K_1 + K_2$, $t = K_2 + K_3$ for $I(K_1, K_2, K_3, K_4)$.

- complicated identities in spinor-helicity language, however, easy in terms of residua in the Grassmanian formulation

Leading singularities

- '60s S-Matrix (Rutgers talk): analytic functional of kinematic invariants [Weinberg][Olive][Chew: Analytic S-Matrix: A Basis for Nuclear Democracy]
- '90s: revival of the idea: unitarity based methods [Bern Dixon Dunbar Kosower][Bern Dixon Kosower]
- recently: highest codimension singularities (leading singularities) [Britto Cachazo Feng][Cachazo]

- leading singularities agree with rational functions C_i
- tree amplitude is a leading singularity as well, two versions related by parity: $A_{\text{BCFW}} = A_{\text{P(BCFW)}}$

$$M_{\text{BCFW}}^{+-+-+-} = (1+g^2+g^4)\left[\frac{\langle 46\rangle^4[13]^4}{[12][23]\langle 45\rangle\langle 56\rangle(p_4+p_5+p_6)^2} \times \frac{1}{\langle 6|5+4|3]\langle 4|5+6|1]}\right]$$

$$M_{\text{P(BCFW)}}^{+-+-+-} = (1+g^2+g^4)\left[\frac{[3|(2+4)|6\rangle^4}{[23][34]\langle 56\rangle\langle 61\rangle(p_5+p_6+p_1)^2} \times \frac{1}{\langle 1|6+5|4]\langle 5|6+1|2]}\right]$$

- completely nontrivial identity in spinor helicity formalism: different formulation?

Grassmanian formulation

The Grassmanian functional [Arkani-Hamed Cachazo Cheung Kaplan]

$$\mathcal{L}_{n;k}(Z_a) = \int \frac{d^{k\times n} C_{\alpha a}}{(12\cdots k)\,(23\cdots(k+1)\,)\,\cdots(n1\cdots(k-1)\,)} \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a} Z_a)\,.$$

$$\alpha = 1..k,\ a = 1...n, \quad \text{minors: } (m_1\cdots m_k) \equiv \varepsilon^{\alpha_1\cdots\alpha_k} C_{\alpha_1 m_1}\cdots C_{\alpha_k m_k}$$

can be brought into a more accessible form

$$\mathcal{L}_{n;k} = \delta^4(\sum_a p_a) J(\lambda,\tilde{\lambda}) \int \frac{d^{(k-2)\times(n-k-2)}\tau}{[(12\cdots k)(23\cdots(k+1))\cdots(n1\cdots(k-1)\,)]\,(\tau)} \prod_I \delta^4(\eta_I + c_{Ii}(\tau)\eta_i)\,,$$

where c_{Ii} are solutions to the kinematical constraints

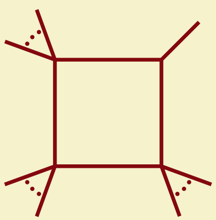
$$\lambda_i - c_{Ii}(\tau)\lambda_I = 0, \quad \tilde{\lambda}_I + c_{Ii}(\tau)\tilde{\lambda}_i = 0.$$

- residue: one complex variable: “$\frac{1}{0}$”
  multiple complex variables: “$\frac{1}{0^{(k-2)\times(n-k-2)}}$”
- Labelling by vanishing minors, i.e. : $\{1,4,7\}$
  totally antisymmetric: $\{i,j,k\}$=-$\{i,k,j\}$
  NMHV-sector: integration variables $\rightarrow n-5$ labels

Grassmanian conjecture

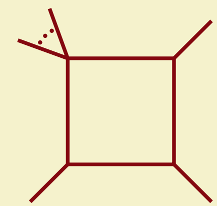
- Conjecture: leading singularities are linear combination of residua
- 3-Mass Box corresponds to exactly one residue [Arkani-Hamed Cachazo Cheung Kaplan]
- other boxes can be obtained from 3-Mass Box [Drummond Henn Korchemsky Sokatchev]

$n = 8,\ k = 3$



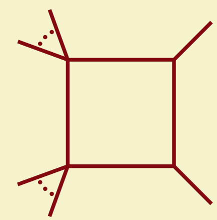
$\{1, 6, 7\}$

$$C^{1\mathrm{m}}_{r,r+1,r+2,r+3} = C^{2me}_{r+2,r+3,r,r+1} + C^{3m}_{r+1,r+2,r+3,r}$$



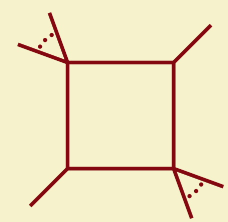
$\{7,1,8\}+\{5,1,8\}$
$+\{5,1,6\}+\{3,1,8\}$
$+\{3,1,6\}+\{3,1,4\}$
$+\{4,5,6\}$

$$C^{2\mathrm{mh}}_{r,r+1,r+2,s} = C^{3m}_{r+1,r+2,s,r} + C^{3m}_{r,r+1,r+2,s}$$



$\{3,4,8\}$
$+\{4,3,5\}$

$$C^{2\mathrm{me}}_{r,r+1,s,s+1} = \sum_{\substack{u,v\\ u\geq r+2\\ u+2\leq v\leq s}} C^{3m}_{r,r+1,u,v} + \sum_{\substack{u,v\\ u\geq s+2\\ u+2\leq v\leq r}} C^{3m}_{s,s+1,u,v}$$



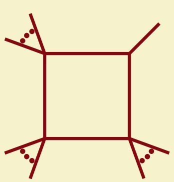
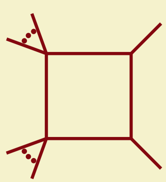
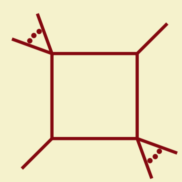
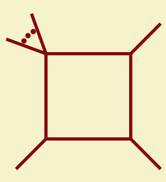
$\{7,1,8\}$
$+\{5,1,8\}$
$+\{3,1,8\}$

Classification of residua

- How to classify residua? *invariant label*: $\{1,3,8\} \overset{n=8}{=} \{251\}$

n	invariant label
7	$\{16\}$,$\{25\}$,$\{34\}$
8	$\{116\}$,$\{125\}$,$\{134\}$,$\{233\}$,$\{224\}$
9	$\{1116\}$,$\{1125\}$,$\{1134\}$,$\{1223\}$,$\{1224\}$,$\{2223\}$
10	$\{11116\}$,$\{11125\}$,$\{11134\}$,$\{11223\}$,$\{11224\}$,$\{12223\}$,$\{22222\}$
⋮	⋮
n	$\{1...16\}$,$\{1...134\}$,$\{1...1233\}$,$\{1...1224\}$,$\{1...12223\}$,$\{1...122222\}$

- NMHV-sector: all leading singularities occur at maximally 3 loops ($n \geq 10$)
  $n = 7$: 1-loop, $n = 8, 9$: 2-loop, [Bullimore Mason Skinner]
- Speculation: even numbers in inv. label: 1⇒1-loop, 3⇒2-loop, 5⇒3-loop
-

3m	2mh	2me	1m
			
$\{125\}$ $\{233\}$	$\{134\}$+$\{116\}$ $\{134\}$+$\{134\}$	$\{116\}$+$\{116\}$ $\{125\}$+$\{134\}$+$\{116\}$	all types

Generalized Residue Theorem (GRT)

- residua are related by multimdimensional version of Cauchy's theorem (GRT):

$$\sum_{j=1}^{n}\{j, i_1, ..., i_{n-6}\} = 0\,.$$

- Conjecture: IR-equations related to GRTs [Arkani-Hamed Cachazo Cheung Kaplan]

Example: $n = 7$, s_{12}:

$$C^{1m}_{1234} + C^{1m}_{7123} + \tfrac{1}{2}C^{2mh}_{1235} - \tfrac{1}{2}C^{2mh}_{6713} + \tfrac{1}{2}C^{2mh}_{1236} - \tfrac{1}{2}C^{2mh}_{3451} - C^{2me}_{7134} - \tfrac{1}{2}C^{3m}_{3461} - \tfrac{1}{2}C^{3m}_{7135}$$

$$(\{7,1\} + \{5,1\} + \{3,1\} + \{4,5\}) + (\{6,7\} + \{4,7\} + \{2,7\} + \{3,4\})$$
$$+\tfrac{1}{2}(\{2,5\} + \{5,6\}) - \tfrac{1}{2}(\{7,3\} + \{3,4\}) + \tfrac{1}{2}(\{2,3\} + \{3,6\}) - \tfrac{1}{2}(\{4,5\} + \{5,1\})$$
$$-(\{7,1\}) - \tfrac{1}{2}(\{3,1\}) - \tfrac{1}{2}(\{7,5\})$$

$$\{2,3\} + \{2,5\} + \{2,7\} + \{4,5\} + \{4,7\} + \{6,7\}$$

General form of the tree-amplitude:

$$M^{\rm tree}_{\rm BCFW} = \mathcal{E} \star \mathcal{O} \star \mathcal{E} \star \cdots$$
$$M^{\rm tree}_{\rm P(BCFW)} = (-1)^n\,\mathcal{O} \star \mathcal{E} \star \mathcal{O} \star \cdots$$

where $\mathcal{E} = 2 + 4 + 6 + \cdots$ and $\mathcal{O} = 1 + 3 + 5 + \cdots$.

One more illustrative example: $n = 9$, s_{1234}:

$$-\frac{1}{2}C^{2mh}_{1235} - \frac{1}{2}C^{2mh}_{5671} + C^{2mh}_{9125} + C^{2mh}_{4561} - \frac{1}{2}C^{2mh}_{8915} - \frac{1}{2}C^{2mh}_{3451} - C^{2me}_{1245} - \frac{1}{2}C^{3m}_{5681} + \frac{1}{2}C^{3m}_{9135}$$
$$+\frac{1}{2}C^{3m}_{4571} + C^{3m}_{1256} - C^{3m}_{5691} + C^{3m}_{9145} + \frac{1}{2}C^{3m}_{1257} + \frac{1}{2}C^{3m}_{4581} + \frac{1}{2}C^{3m}_{1258} + \frac{1}{2}C^{3m}_{5613} - \frac{1}{2}C^{3m}_{9157} = 0\,.$$

translates (without using GRTs) into

$$\begin{aligned}\frac{1}{2}\,(&-\{1,2,3,5\} - \{1,2,3,7\} - \{1,2,4,5\} - \{1,2,4,7\} \\ &+ \{1,2,5,6\} + \{1,2,5,8\} + \{1,2,5,9\} - \{1,2,6,7\} \\ &+ \{1,2,7,8\} + \{1,2,7,9\} - \{1,5,6,7\} - \{2,5,6,7\} \\ &-\{3,5,6,7\} - \{4,5,6,7\} + \{5,6,7,8\} + \{5,6,7,9\}) = 0\end{aligned}$$

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$$+\frac{1}{2}C^{3m}_{4571} + C^{3m}_{1256} - C^{3m}_{5691} + C^{3m}_{9145} + \frac{1}{2}C^{3m}_{1257} + \frac{1}{2}C^{3m}_{4581} + \frac{1}{2}C^{3m}_{1258} + \frac{1}{2}C^{3m}_{5613} - \frac{1}{2}C^{3m}_{9157} = 0\,.$$

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$$\begin{aligned}\frac{1}{2}\big(&-\{1,2,3,5\} - \{1,2,3,7\} - \{1,2,4,5\} - \{1,2,4,7\} \\ &+\{1,2,5,6\} + \{1,2,5,8\} + \{1,2,5,9\} - \{1,2,6,7\} \\ &+\{1,2,7,8\} + \{1,2,7,9\} - \{1,5,6,7\} - \{2,5,6,7\} \\ &-\{3,5,6,7\} - \{4,5,6,7\} + \{5,6,7,8\} + \{5,6,7,9\}\big) = 0\end{aligned}$$

$$(1,2,5) + (1,2,7) + (5,6,7) = 0$$

Mapping IR-equations to general residue theorems

particles	kin. inv	sources
7	s_{123}	$0=(1)$
8	s_{123}	$0=(1,4)+(1,6)$
	s_{1234}	$0=(1,2)+(5,6)$
9	s_{123}	$0=(1,4,5)+(1,4,7)+(1,6,7)$
	s_{1234}	$0=(1,2,5)+(1,2,7)+(5,6,7)$
10	s_{123}	$0=(1,4,5,6)+(1,4,5,8)+(1,4,7,8)+(1,6,7,8)$
	s_{1234}	$0=(1,2,5,6)+(1,2,5,8)+(1,2,7,8)+(5,6,7,8)$
	s_{12345}	$0=(1,2,3,6)+(1,2,3,8)+(1,6,7,8)+(3,6,7,8)$
$\vdots$	$\vdots$	$\vdots$
12	s_{123}	$0=(1,4,5,6,7,8)+(1,4,5,6,7,10)+(1,4,5,6,9,10)$ $+(1,4,5,8,9,10)+(1,4,7,8,9,10)+(1,6,7,8,9,10)$
	s_{1234}	$0=(1,2,5,6,7,8)+(1,2,5,6,7,10)+(1,2,5,6,9,10)$ $+(1,2,5,8,9,10)+(1,2,7,8,9,10)+(5,6,7,8,9,10)$
	s_{12345}	$0=(1,2,3,6,7,8)+(1,2,3,6,7,10)+(1,2,3,6,9,10)$ $+(1,2,3,8,9,10)+(1,6,7,8,9,10)+(3,6,7,8,9,10)$
	s_{123456}	$0=(1,2,3,4,7,8)+(1,2,3,4,7,10)+(1,2,3,4,9,10)$ $+(1,2,7,8,9,10)+(1,4,7,8,9,10)+(3,4,7,8,9,10)$

Mapping IR-equations to general residue theorems

For s_{123}, the general rule reads:

$$1\star(4+6)\star(5+7)\star\cdots\star((n-4)+(n-2))=0\,,$$

while for $s_{12\ldots m}$, $m>3$:

$$\begin{aligned}(1,2,\ldots,m{-}2)\star((m{+}1)+(m{+}3))\star((m{+}2)+(m{+}4))\star\cdots\star((n-4)+(n-2))\\ +(1{+}3)\star\cdots\star((m{-}5)+(m{-}3))\star((m{-}4)+(m{-}2))\star((m{+}1),\ldots,(n{-}3),(n{-}2))=0,\end{aligned}$$

where all others follow from cyclic invariance.

"Remarkable" identities...

$$M_{\text{BCFW}} = M_{\text{P(BCFW)}}$$
$$\mathcal{E} \star \mathcal{O} \star \mathcal{E} \star \cdots = (-1)^{(n-5)} \mathcal{O} \star \mathcal{E} \star \mathcal{O} \star \cdots$$

can be obtained from adding GRTs with *all* sourceterms of the form oe:

$(1,2)$, $(1,4)$, $(1,6)$, $(1,8)$, $(3,4)$, $(3,6)$, $(3,8)$, $(5,6)$, $(5,8)$ und $(7,8)$

Example for $n = 8$:

$$\begin{aligned}
&(\{2,3,4\} + \{2,3,6\} + \{2,3,8\} + \{2,5,6\} + \{2,5,8\} \\
&\quad + \{2,7,8\} + \{4,5,6\} + \{4,5,8\} + \{4,7,8\} + \{6,7,8\}) \\
&\quad = -(\{1,2,3\} + \{1,2,5\} + \{1,2,7\} + \{1,4,5\} + \{1,4,7\} \\
&\qquad + \{1,6,7\} + \{3,4,5\} + \{3,4,7\} + \{3,6,7\} + \{5,6,7\})
\end{aligned}$$

Parity-invariance: same result from adding all sourceterms eo.

Outlook and open questions

- nicely accessible formulation of one-loop IR-equations in terms of GRTs (only NMHV-sector)
- a good language for N^pMHV, $p > 1$ has to be found - Plücker-labeling not sufficient
- $\binom{n}{n-5}$ residua are still redundant - after employing all GRTs, the minimal set has dimension $\binom{n-1}{n-5}$
- for the NMHV-sector, the tree-amplitude is the only non-IR-divergent quantity: all other divergences should be related to it. Is it possible to retrieve information about higher-loop IR-divergences? General structure studied in [Anastasiou Bern Dixon Kosower]
- Integral basis for higher-loop amplitudes?

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THANKS !