Higher spins of mixed symmetry

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A. C., D. Francia, J. Mourad and A. Sagnotti, 0810.4350, 0904.4447

For a review see A. C., 0905.1472, 0910.3155

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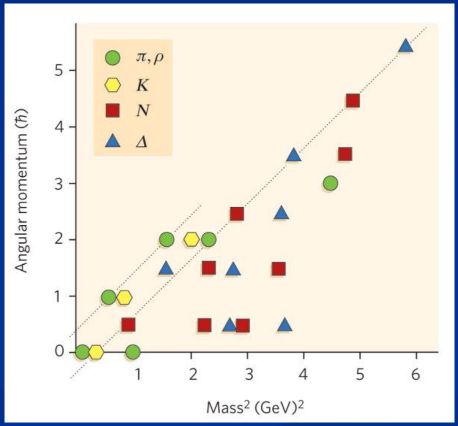
Why higher spins?

- Long standing open questions in field theory (see for instance Fierz-Pauli, 1939)

- Another setup where higher spins play a key role: String Theory



- String Theory was indeed designed to describe ∞ higher-spin resonances

- What can we learn from String Theory?

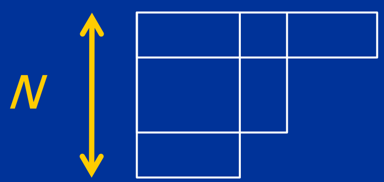
- What can we add to the understanding of String Theory?
  - Example: String Theory was conjectured to be a broken phase of an higher-spin gauge theory. How to make precise this statement?

String Theory and mixed symmetry

- String models are consistent in $D > 4$
- If $D > 4$ the little groups $SO(D-1)$ or $SO(D-2)$ admit more than a single Casimir operator
- Their irreps are characterized by sets of labels $\{ s_1, \ldots , s_N \}$ with $N \leq (D-1)/2$ or $N \leq (D-2)/2$
- $N > 1 \rightarrow$ Mixed symmetry representations
- Mixed symmetry fields enter all String spectra
- Challenging setup for massless fields:
  - Symmetric fields ($N=1$): Vasiliev full non-linear theory in AdS
  - No known extensions of the Vasiliev system for $N>1$

Mixed-symmetry fields

- Free particles in D = 4
  - Mass and spin
- Free particles in D > 5
  - Irreps of SO(D-2)



- Fully symmetric fields

$$\varphi_{\mu_1 \dots \mu_s} \qquad \psi^{\alpha}{}_{\mu_1 \dots \mu_s}$$

- Mixed-symmetry fields

$$\varphi_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots}$$

$$\psi^{\alpha}{}_{\underbrace{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots}_{N}}$$

- In String Theory

$$\varphi_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots} \quad \alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_{s_1}} \; \alpha_{-2}^{\nu_1} \dots \alpha_{-2}^{\nu_{s_2}} \; \dots \, | \, 0 \, \rangle$$

- **How to write field equations and Lagrangians?**

Outline

- Motivations
- Constrained free theory for massless fields (metric-like)
  - Non-Lagrangian field equations for Bose and Fermi fields
  - Lagrangians for Bose and Fermi fields
- Comments on the role of the constraints
- Minimal unconstrained local theory for massless fields
  - Non-Lagrangian field equations & auxiliary fields
  - Lagrangians & auxiliary fields
- Weyl-like symmetries
- Conclusions

Massless fields & gauge symmetries

- M = 0 $\rightarrow$ gauge fields
  - Gravity $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$
  - Fully symmetric fields $\delta \varphi_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \Lambda_{\mu_2 \dots \mu_s)}$
  - Mixed-symmetry fields

$$\delta \varphi_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots} = \partial_{(\mu_1} \Lambda^{(1)}{}_{\mu_2 \dots \mu_{s_1}), \nu_1 \dots \nu_{s_2}, \dots} + \partial_{(\nu_1 |} \Lambda^{(2)}{}_{\mu_1 \dots \mu_{s_1}, | \nu_2 \dots \nu_{s_2}), \dots} + \dots ,$$

- A good notation is needed...
  - Hide the space-time indices $\varphi_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots} \rightarrow \varphi$
  - Select the "family" where the operators act $\partial_{(\mu_1} \Lambda^{(1)}{}_{\mu_2 \dots \mu_{s_1}), \nu_1 \dots \nu_{s_2}, \dots} \rightarrow \partial^1 \Lambda_1$

- Gauge transformations: $\boxed{\delta \varphi = \partial^i \Lambda_i}$ $\boxed{\delta \psi = \partial^i \epsilon_i}$

Field equations

- An example: linearized gravity!

$$R_{\mu\nu} = \Box h_{\mu\nu} - (\partial_\mu \partial \cdot h_\nu + \partial_\nu \partial \cdot h_\mu) + \partial_\mu \partial_\nu h^\lambda{}_\lambda = 0$$

- Labastida field equations — *Fronsdal, Fang (1978)* — *Labastida (1986)*
  - Bose $$\mathcal{F} \equiv \Box \varphi - \partial^i \partial_i \varphi + \frac{1}{2} \partial^i \partial^j T_{ij} \varphi = 0$$
  - Fermi $$\mathcal{S} \equiv i \left(\not{\partial} \psi - \partial^i \gamma_i \psi \right) = 0$$

- Gauge variation
  - Bose $$\delta \mathcal{F} = \frac{1}{2} \partial^i \partial^j \partial^k T_{(ij} \Lambda_{k)}$$
  - Fermi $$\delta \mathcal{S} = -\frac{i}{2} \partial^i \partial^j \gamma_{(i} \epsilon_{j)}$$

- Gauge invariant field equations → constraints on Λ_i, ϵ_i

Again the gravity example…

- Non-Lagrangian field equation

$$R_{\mu\nu} = \Box h_{\mu\nu} - (\partial_\mu \partial \cdot h_\nu + \partial_\nu \partial \cdot h_\mu) + \partial_\mu \partial_\nu h^\lambda{}_\lambda = 0$$

- "Natural ansatz" for the Lagrangian

$$\mathcal{L} = \frac{1}{2} h^{\mu\nu} \left(R_{\mu\nu} + k\, \eta_{\mu\nu} R_\lambda{}^\lambda \right),$$

- Gauge variation

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad \Rightarrow \quad \delta \mathcal{L} = -\, \xi^\mu \left(\partial^\lambda R_{\mu\lambda} + k\, \partial_\mu R_\lambda{}^\lambda \right),$$

- Bianchi identity

$$\partial^\lambda R_{\mu\lambda} - \frac{1}{2} \partial_\mu R_\lambda{}^\lambda = 0 \qquad \Rightarrow \qquad k = -\frac{1}{2}$$

Lagrangians for arbitrary fields

- Bianchi identities
  - Bose $$\partial_i \mathcal{F} - \frac{1}{2}\partial^j T_{ij}\mathcal{F} = -\frac{1}{12}\partial^j\partial^k\partial^l T_{(ij}T_{kl)}\varphi$$
  - Fermi $$\partial_i \mathcal{S} - \frac{1}{2}\not\partial \not\mathcal{S}_i - \frac{1}{2}\partial^j T_{ij}\mathcal{S} - \frac{1}{6}\partial^j\gamma_{ij}\mathcal{S} = \frac{i}{6}\partial^j\partial^k T_{(ij}\gamma_{k)}\psi$$
- They lead to the constraints

$$T_{(ij}T_{kl)}\varphi = 0 \qquad T_{(ij}\gamma_{k)}\psi = 0$$

- Natural ansatz for the Lagrangians

Labastida (1986)

$$\mathcal{L}_{\text{Bose}} = \frac{1}{2}\langle\varphi, \sum_{p=0}^{N} k_p\,\eta^p\mathcal{F}^{[p]}\rangle$$

A.C., Francia, Mourad, Sagnotti (2009)

$$\mathcal{L}_{\text{Fermi}} = \frac{1}{2}\langle\bar{\psi}, \sum_{p,q=0}^{N} k_{p,q}\,\eta^p\gamma^q(\gamma^{[q]}\mathcal{S}^{[p]})\rangle + \text{h.c.}$$

Constrained Lagrangians

- Bose fields

Labastida (1989)

$$\mathcal{L}_{\text{Bose}} = \frac{1}{2}\langle \varphi , \sum_{p=0}^{N} \frac{(-1)^{p}}{p!\,(p+1)!}\, \eta^{p}\, \mathcal{F}^{[p]} \rangle$$

- Fermi fields

A.C., Francia, Mourad, Sagnotti (2009)

$$\mathcal{L}_{\text{Fermi}} = \frac{1}{2}\langle \bar{\psi} , \sum_{p,q=0}^{N} \frac{(-1)^{p+\frac{q(q+1)}{2}}}{p!\,q!\,(p+q+1)!}\, \eta^{p}\gamma^{q}\,(\gamma^{[q]}\mathcal{S}^{[p]})\rangle + \text{h.c.}$$

- Constrained because...

$$\delta\,\mathcal{F} = \frac{1}{2}\,\partial^{i}\partial^{j}\partial^{k}\, T_{(ij}\Lambda_{k)}$$

$$\delta\,\mathcal{S} = -\frac{i}{2}\,\partial^{i}\partial^{j}\,\gamma_{(i}\,\epsilon_{j)}$$

$$\partial_{i}\,\mathcal{F} - \frac{1}{2}\,\partial^{j}\,T_{ij}\,\mathcal{F} = -\frac{1}{12}\,\partial^{j}\partial^{k}\partial^{l}\,T_{(ij}\,T_{kl)}\,\varphi$$

$$\partial_{i}\,\mathcal{S} - \frac{1}{2}\,\not{\partial}\,\not{\mathcal{S}}_{i} - \frac{1}{2}\,\partial^{j}T_{ij}\,\mathcal{S} - \frac{1}{6}\,\partial^{j}\gamma_{ij}\,\mathcal{S} = \frac{i}{6}\,\partial^{j}\partial^{k}\,T_{(ij}\gamma_{k)}\,\psi$$

Comments on the constraints

- These constraints should not obstruct a possible non-linear deformation of the free theory
  - The Vasiliev theory contains constrained fields

    Fradkin, Vasiliev (1987)

    Vasiliev (1990-2010)

- Still, they are quite unsatisfactory…
  - No constraints in the massless limit of free String Field Theory

    Henneaux, Teitelboim (1988), Sagnotti, Tsulaia (2004)
  - The constrained theory cannot be related to the linearized curvatures of de Wit and Freedman *De Wit, Freedman(1980)*

- Higher spin geometry: $\delta\,\Gamma_{\lambda\,,\,\mu\nu} = \partial_\mu\,\partial_\nu\,\xi_\lambda$

$$\delta\,\Gamma^{(k)}{}_{\lambda_1\dots\,\lambda_k\,,\,\mu_1\dots\,\mu_s} = \partial_{(\,\mu_1} \dots \partial_{\mu_{k+1}} \Lambda_{\mu_{k+2}\dots\,\mu_s)\,\lambda_1\dots\,\lambda_k}$$

- It is possible to use the HS curvatures to obtain non-local Lagrangians describing the correct local dynamics

  Francia, Sagnotti, Mourad (2002-2010), Bekaert, Boulanger (2002), De Medeiros, Hull (2003)

Unconstrained theory

- Key steps in constructing the constrained theory
  - Gauge invariance → kinetic tensors & constraints on the gauge parameters
  - Bianchi identities → constraints on the fields
  - Traces of the Bianchi identities → uniquely fix the Lagrangians

- In the unconstrained theory
  - Gauge invariance → auxiliary compensator fields
  - Bianchi identities → Lagrange multipliers
  - Traces of the Bianchi identities → fix the Lagrangians up to field redefinitions

Gauge invariance & auxiliary fields

- Gauge variation of the kinetic tensors
  - Bose $\delta\,\mathcal{F} = \frac{1}{2}\,\partial^{\,i}\partial^{\,j}\partial^{\,k}\,T_{(\,ij}\Lambda_{\,k\,)}$
  - Fermi $\delta\,\mathcal{S} = -\,\frac{i}{2}\,\partial^{\,i}\partial^{\,j}\,\gamma_{\,(\,i}\,\epsilon_{\,j\,)}$

Francia, Sagnotti (2002, 2005)
Francia, Mourad, Sagnotti (2007)
A.C., Francia, Mourad, Sagnotti (2008)

- Compensator fields
  - Bose $\delta\,\Phi_{\,i} = \Lambda_{\,i} \;\rightarrow\; \mathcal{A} = \mathcal{F} - \frac{1}{6}\,\partial^{\,i}\partial^{\,j}\partial^{\,k}\,T_{(\,ij}\Phi_{\,k\,)}$
  - Fermi $\delta\,\Psi_{\,i} = \epsilon_{\,i} \;\rightarrow\; \mathcal{W} = \mathcal{S} + \frac{i}{2}\,\partial^{\,i}\partial^{\,j}\,\gamma_{\,(\,i}\,\Psi_{\,j\,)}$

- Eliminating the auxiliary fields one recovers the non-local field equations of Francia and Sagnotti *Francia (2010)*

Biianchi identities & auxiliary fields

- Unconstrained Bianchi identities

$$\partial_i\,\mathcal{A} - \frac{1}{2}\,\partial^j\,T_{ij}\,\mathcal{A} \;=\; -\,\frac{1}{12}\,\partial^j\partial^k\partial^l\,T_{(\,ij}\,T_{kl\,)}\,(\,\varphi - \partial^{\,m}\,\Phi_{\,m}\,) \qquad \mathcal{C}_{ijkl}$$

$$\partial_i\,\mathcal{W} - \frac{1}{2}\,\not\partial\,\not\mathcal{W}_i - \frac{1}{2}\,\partial^j T_{ij}\,\mathcal{W} - \frac{1}{6}\,\partial^j\gamma_{ij}\,\mathcal{W} \;=\; \frac{i}{6}\,\partial^j\partial^k\,T_{(\,ij}\gamma_{\,k\,)}\,(\,\psi - \partial^{\,m}\,\Psi_{\,m}\,) \qquad \mathcal{Z}_{ijk}$$

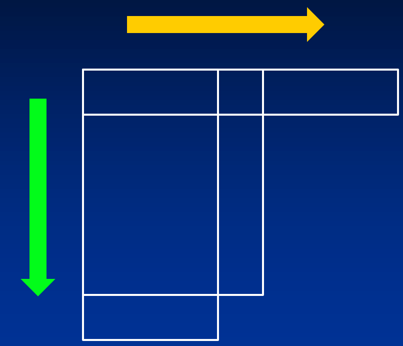
- Lagrange multipliers to compensate the remainder

$$\mathcal{L}_{\,\mathrm{Bose}} \;=\; \frac{1}{2}\,\langle\,\varphi - \partial^{\,m}\Phi_{\,m}\,,\,\sum_{p\,=\,0}^{N}\frac{(-1)^p}{p!(p+1)!}\,\eta^{\,p}\,\mathcal{A}^{\,[\,p\,]}\,\rangle + \frac{1}{8}\,\langle\,\beta_{\,ijkl}\,,\,\mathcal{C}_{ijkl}\,\rangle$$

$$\mathcal{L}_{\,\mathrm{Fermi}} \;=\; \frac{1}{2}\,\langle\,\bar{\psi} - \partial^{\,m}\bar{\Psi}_{\,m}\,,\,\sum_{p,q\,=\,0}^{N}\frac{(-1)^{p+\frac{q(q+1)}{2}}}{p!q!(p+q+1)!}\,\eta^{\,p}\gamma^{\,q}\,(\,\gamma^{\,[\,q\,]}\mathcal{W}^{\,[\,p\,]}\,)\,\rangle$$

$$+\,\frac{1}{4}\,\langle\,\bar{\lambda}_{\,ijk}\,,\,\mathcal{Z}_{\,ijk}\,\rangle + \mathrm{h.c.}$$

Structure of the Lagrangians



- Two peculiar features:
  - Compensators and Lagrange multipliers
  - Higher traces or γ-traces of the kinetic tensor in the Lagrangians

$$\mathcal{L}_{\text{Bose}} = \frac{1}{2}\langle \varphi - \partial^m \Phi_m \, , \sum_{p=0}^{N} \frac{(-1)^p}{p!(p+1)!}\,\eta^p \mathcal{A}^{[p]} \rangle + \frac{1}{8}\langle \beta_{ijkl} \, , \mathcal{C}_{ijkl} \rangle$$

$$\mathcal{L}_{\text{Fermi}} = \frac{1}{2}\langle \bar{\psi} - \partial^m \bar{\Psi}_m \, , \sum_{p,q=0}^{N} \frac{(-1)^{p+\frac{q(q+1)}{2}}}{p!q!(p+q+1)!}\,\eta^p \gamma^q (\gamma^{[q]} \mathcal{W}^{[p]}) \rangle + \frac{1}{4}\langle \bar{\lambda}_{ijk} \, , \mathcal{Z}_{ijk} \rangle + \text{h.c.}$$

Higher spins

Mixed symmetry

Equations of motion

- Field equations for bosons

$$\sum_{p=0}^{N} \frac{(-1)^{p}}{p!\,(p+1)!}\,\eta^{p}\,\mathcal{F}^{[p]} + \frac{1}{2}\,\eta^{ij}\,\eta^{kl}\,\mathcal{B}_{ijkl} = 0$$

$$T_{(ij}\,T_{kl)}\,\mathcal{F} = 0$$

- Field equations for fermions

$$\sum_{p,q=0}^{N} \frac{(-1)^{p+\frac{q(q+1)}{2}}}{p!\,q!\,(p+q+1)!}\,\eta^{p}\gamma^{q}\,(\gamma^{[q]}\mathcal{S}^{[p]}) - \frac{1}{2}\,\eta^{ij}\,\gamma^{k}\,\mathcal{Y}_{ijk} = 0$$

$$T_{(ij}\,\gamma_{k)}\,\mathcal{S} = 0$$

- Are they equivalent to $\mathcal{F} = 0$ and $\mathcal{S} = 0$?
  YES if D $\geq$ 2(N+1) but...

Possible pathologies

- Recall 2D gravity

$$R_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,R = 0 \qquad \not\Rightarrow \qquad R_{\mu\nu} = 0$$

- In general what can happen?
  - $\mathcal{F}$ or $\mathcal{S}$ do not vanish but the Lagrangians do
    - Two-dimensional graviton & gravitino
    - Two-column bosons and single-column fermions in low enough space-time dimensions
  - The Lagrangians possess more symmetries with respect to the non-Lagrangian field equations
    - Some classes of mixed-symmetry bosons and fermions for D < 2(N+1)
- Less equations but new symmetries

$$\delta\,\mathcal{F} = \eta^{\,ij}\,\Theta_{\,ij} \qquad\qquad \delta\,\mathcal{S} = \gamma^{\,i}\,\Theta_{\,i}$$

The classification

A.C., Francia, Mourad, Sagnotti (2008-2009)

- $[\, T_{ij} \,, \eta^{\,kl} \,] \;=\; \frac{D}{2}\, \delta_{i}{}^{(k}\, \delta_{j}{}^{l)} \,+\, \frac{1}{2} \Big(\delta_{i}{}^{(k} S^{l)}{}_{j} \,+\, \delta_{j}{}^{(k} S^{l)}{}_{i} \Big)$
- $S^{i}{}_{j}$ operators: $S^{1}{}_{2}\, \varphi \;\rightarrow\; \varphi_{(\,\mu_1 \dots \mu_{s_1}\,,\, \mu_{s_1+1}\,)\, \nu_1 \dots\, \nu_{s_2-1}}\,, \dots$
- Considering $\delta \mathcal{F} \,=\, \eta^{\,ij}\, \Theta_{ij}$ and $\delta \mathcal{S} \,=\, \gamma^{\,i}\, \Theta_{i}$ leads to
  - Bose $\quad (\, D - 2\,)\, \Theta_{\,ij} \,+\, S^{\,k}{}_{(\,i}\, \Theta_{\,j\,)\,k} \;=\; 0$
  - Fermi $\quad (\, D - 2\,)\, \Theta_{\,i} \,+\, 2\, S^{\,j}{}_{i}\, \Theta_{\,j} \;=\; 0$
- Classification of Weyl-like symmetries

  ↕

  eigenvalue problems for the $S^{\,i}{}_{j}$ operators

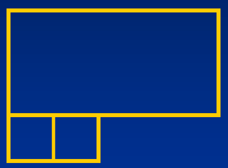
$$[\, S^{i}{}_{j} \,,\, S^{k}{}_{l} \,] \;=\; \delta_{j}{}^{k} S^{i}{}_{l} \,-\, \delta_{l}{}^{i} S^{k}{}_{j}$$

⇐ gl(N) algebra

Weyl invariant models

A.C., Francia, Mourad, Sagnotti (2009)

- Examples of solutions of the previous conditions
  - Bose $\{2^{N-1}, 2\}-$ projected fields in D = 2N
  - Fermi $\{2^{N-1}, 1\}-$ projected fields in D = 2N



- The Weyl-like shifts also preserve
  - Bose $T_{(ij}T_{kl)}\mathcal{F} = 0$
  - Fermi $T_{(ij}\gamma_{k)}\mathcal{S} = 0$
- Other Weyl-invariant models
  - Bose $\{2^p, 1^q\}-$ projected fields in D ≤ 2p+q
  - Fermi $\{1^q\}-$ projected fields in D ≤ 2q

- In all these cases no local d.o.f. are involved

Summary & outlook

- Simple guidelines
  - Gauge invariance
  - Bianchi identities
- New results
  - Completion of the constrained theory for Fermi fields
    - Check of the propagating d.o.f.
    - Lagrangians
  - Unconstrained Lagrangians for Bose and Fermi fields
  - Weyl-like symmetries
- Perspectives
  - Extension to (A)dS, Supersymmetry
  - Links with String Theory
  - Interactions…