Exact semiclassical strings

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Motivation
I. general aspects

- Suggestion of AdS/CFT (N=4 SYM in 4-dim ⟷ Type II B strings in $AdS_5 x S^5$)

  > understanding quantum gauge theories at any coupling
  > understanding string theories in non-trivial backgrounds

  better address them together than separately.

- Fundamental insight: both theories might be integrable.
  Many efforts to translate into solvable.
  Solvability even beyond the spectrum.

  [Bena, Polchinski, Roiban, 02]
  [Minahan Zarembo 02]
  [Beisert, Kristjansen, Staudacher 03]
  [Beisert, Staudacher 05]
  [Beisert, Henn, McLaughlin, Plefka 10]
  [Alday, Maldacena, Sever, Vieira 10]

- Recent proposals to solve their full planar spectrum need explicit and independent checks.

  [Gromov, Kazakov, Vieira, 09]
  [Arutyunov, Frolov 09]
  [Bombardelli, Fioravanti, Tateo 09]

- Isometries of the two models coincide: $PSU(2,2|4)$
  Gauge theory operators & string states organized in reprs -$(E; S_1, S_2; J_1, J_2, J_3)$

- String energies = dimensions of dual gauge operators

$$E(\sqrt{\lambda}, C, ...) \equiv \Delta(\lambda, C, ...) \qquad C = (S_1, S_2; J_1, J_2, J_3)$$

Motivation
II. our aim

- Crucial role of *semiclassical* quantization of strings [Gubser, Klebanov, Polyakov 02]

  and crucial example: *folded* string rotating fastly (with spin S) in AdS_3

> Semiclassical expansion $(\lambda \gg 1$ & $\mathcal{S} = S/\sqrt{\lambda}$ finite $)$ & $\mathcal{S} \gg 1$
>
> $$E = S + f(\lambda)\ \ln S + \ldots \qquad f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi}\Big[a_0 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + \ldots\Big]$$
>
> [Frolov Tseytlin 02] [Roiban Tseytlin 07]
>
> exactly confirmed by solution of integrability-based "cusp anomaly" equation
>
> [Beisert, Eden, Staudacher 06]
> [Basso, Korchemsky, Kotansky]
>
> $$\gamma(S) = f(\lambda)\ \log S + O(S^0) \qquad \mathcal{O}_S = \mathrm{Tr}(\varphi D^S \varphi) \qquad S \gg 1$$
>
> checked at weak coupling by MHV 4-point gluon amplitudes.
>
> [Bern, Czakon, Dixon, Kosower, Smirnov, 06]

- In order to
  - > go (safely) beyond the leading logarithmic behavior
  - > extrapolate data for operators with finite quantum numbers

  one needs **additional analytic tools** on the string side, from which better understanding of quantum corrections for strings in AdS_5xS^5.

Outlook

The setup

- ***GS folded string***

The 1-loop "exact" calculation

- ***Gauge-related issues***
- ***Solvable structures and solution***

Results useful in AdS/CFT

- ***Long strings***
- ***Short strings***

Summary

Setup I: GS superstrings in $AdS^5 x S_5$

[Metsaev, Tseytlin 98]

$$I = -\frac{\sqrt{\lambda}}{2\,\pi} \int d^2\xi \left[L_B(x,y) + L_F(x,y,\theta) \right]$$

$$\frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'} = T$$ string tension

$$L_B = \frac{1}{2}\sqrt{-g}g^{ab}\left[G^{(AdS_5)}_{mn}(x)\,\partial_a x^m \partial_b x^n + G^{(S^5)}_{m'n'}(y)\partial_a y^{m'}\partial_b y^{n'} \right]$$

$$L_F = i(\sqrt{-g}g^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})\,\bar{\theta}^I\,\rho_a\,D_b\,\theta^J + \mathcal{O}(\theta^4)$$

> $\hbar \leftrightarrow 1/\sqrt{\lambda}$ semiclassical expansion in powers of $1/\sqrt{\lambda}$

$$E = \sqrt{\lambda}\,\mathcal{E} = \sqrt{\lambda}\left[\mathcal{E}_0 + \frac{\mathcal{E}_1}{\sqrt{\lambda}} + \frac{\mathcal{E}_2}{(\sqrt{\lambda})^2} + ... \right]$$

> invariant under world-sheet diffeo and local fermionic *k*-symmetry

Setup II: folded string in AdS_3

- Ansatz for a solution rotating in AdS_3

$$\rho = \rho(\sigma) \qquad t = \kappa\,\tau \qquad \phi = \omega\,\tau$$

> Sinh-Gordon EOM for $\rho(\sigma)$

$$\rho'' = \frac{1}{2}(\kappa^2 - \omega^2)\,\sinh(2\rho)$$

> Conformal gauge constraint

$$\rho'^2 = \kappa^2\,\cosh^2\rho - \omega^2\,\sinh^2\rho \qquad\qquad 0 \le \rho(\sigma) < \rho_{\max}$$

$$\coth\rho_{max} = \frac{\omega}{\kappa} \equiv \sqrt{1 + \frac{1}{\epsilon^2}}$$

ϵ ~ length of the string

- Exact solution $\quad \sinh\rho = \epsilon\,\mathrm{sn}\left[\frac{\kappa\,\sigma}{\epsilon}, -\epsilon^2\right], \quad 0 \le \sigma < \frac{\pi}{2}$

Setup III: folded string, classical results

▶ Integrals of motion: classical energy and spin

$$E = P_t = \sqrt{\lambda}\,\kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2\rho \equiv \sqrt{\lambda}\,\mathcal{E} \qquad\qquad S \;=\; P_\phi = \sqrt{\lambda}\,\omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2\rho \equiv \sqrt{\lambda}\,\mathcal{S}$$

In parametric form

$$\mathcal{E} = \frac{E}{\sqrt{\lambda}} = \frac{2\epsilon}{\pi}\,\mathbb{E}(-\epsilon^2), \qquad \mathcal{S} = \frac{S}{\sqrt{\lambda}} = \frac{2\sqrt{1+\epsilon^2}}{\pi}\left[\mathbb{E}(-\epsilon^2) - \mathbb{K}(-\epsilon^2)\right]$$

> Long strings (large spin) $\epsilon \to +\infty$ [Gubser, Klebanov, Polyakov 02]

$$S \gg \sqrt{\lambda} \qquad\qquad E \sim \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}}$$

as twist operators at weak coupling

> Short strings (small spin) $\epsilon \to 0$ [Gubser, Klebanov, Polyakov 98]

$$S \ll \sqrt{\lambda} \qquad\qquad E \sim \sqrt{2\sqrt{\lambda}\,S}$$

flat space $\sqrt{\lambda} \sim \frac{1}{\alpha'}$

Semiclassical quantization of folded string

- Standard quantization of a soliton

> Background field method

$$\mathcal{L} \xrightarrow{\phi = \phi_{cl} + \frac{\tilde{\phi}}{\sqrt[4]{\lambda}}} \tilde{\mathcal{L}}_{\text{fluct}}$$

> Effective action

$$\Gamma = -\ln Z = -\ln \frac{\det \text{fermions}}{\sqrt{\det \text{bosons}}}$$

> 1-loop energy

$$E_1 = \frac{\Gamma_1}{\kappa \mathcal{T}}\,, \qquad \mathcal{T} \equiv \int d\tau \to \infty$$

- Stationary solution → 1-dimensional determinants!

$$\det\left[-\partial_\tau^2 - \partial_\sigma^2 + M^2(\sigma)\right] = \mathcal{T} \int \frac{d\omega}{2\pi} \left[-\partial_\sigma^2 + \omega^2 + M^2(\sigma)\right]$$

Quantum fluctuations: Fermions

[Drukker,Gross, Tseytlin 00]

[Frolov, Tseytlin 02]

$$\mathcal{L}_{2F}^{GS} \xrightarrow[SO(1,9)\ \text{rotation}]{\text{k-symmetry fix.}\quad \theta^1=\theta^2} \mathcal{L}_{2F} = 2\,\bar{\theta}\,D_F\,\theta$$

$$D_F = i(\Gamma^a\,\partial_a - \mu_F\,\Gamma_{234}), \qquad \mu_F = \rho'(\sigma)$$

- Determinant of the "diagonalized" laplacian

$$\ln\det D_F = \frac{1}{2}\ln\det(D_F)^2 \longrightarrow \frac{1}{2}\Big[4\ln\det\Delta_{F_+} + 4\ln\det\Delta_{F_-}\Big]$$

$>$ 8 species of Majorana fermions in 2 dimensions

$$\Delta_{F_\pm} = -\partial_a\partial^a + \hat{\mu}^2_{F_\pm} \qquad\qquad \hat{\mu}^2_{F_\pm} = \pm\rho'' + \rho'^2$$

- Conformal gauge ghosts and k-symmetry ghosts decouple.

Quantum fluctuations: Bosons

1. Conformal gauge : nasty AdS_3 coupled sector

$$\mathcal{L}^{\rm conf}_{\rm 2B} = -\partial_a \tilde{t}\partial^a\tilde{t} - \mu^2_{\tilde{t}}\,\tilde{t}^2 + \partial_a\tilde{\rho}\partial^a\tilde{\rho} + \mu^2_{\tilde{\rho}}\,\tilde{\rho}^2 + \partial_a\tilde{\phi}\partial^a\tilde{\phi} + \mu^2_{\tilde{\phi}}\,\tilde{\phi}^2 + \partial_a\tilde{\beta}_i\partial^a\tilde{\beta}_i + \mu^2_{\beta_i}\tilde{\beta}_i^2 + $$
$$+4\,\tilde{\rho}\Big(k\,\sinh\rho\,\partial_\tau\tilde{t} - \cosh\rho\,\partial_\tau\tilde{\phi}\Big)$$

$\mu^2_{\tilde{t}} = 2\rho'^2 - \kappa^2$ etc.

> highly non trivial masses! but regular

> 3 massive and coupled AdS_3 fluctuations $\tilde{t},\ \tilde{\rho},\ \tilde{\phi}$

$$Q(\sigma) = \begin{pmatrix} \partial_\sigma^2 - \omega^2 - \rho'^2 + \kappa^2 & 2\,\omega\,\kappa\,\sinh\rho & 0 \\ -2\,\omega\,\kappa\,\sinh\rho & -\partial_\sigma^2 + \omega^2 + 2\rho'^2 - \omega_p^2 & 2\,\omega\,\omega_p\,\cosh\rho \\ 0 & -2\,\omega\,\omega_p\,\cosh\rho & -\partial_\sigma^2 + \omega^2 + \rho'^2 - \omega_p^2 - \kappa^2 \end{pmatrix}$$

Determinant of $Q(\sigma)$ - hard to find in closed form → expansion needed.

2. Static gauge ($\tilde\rho = \tilde t = 0$) :

$$\mathcal{L}^{\rm st}_{\rm 2B} \;=\; \partial_a\bar\phi\partial^a\bar\phi + m^2_{\bar\phi} + \partial_a\,\tilde\beta_i\partial^a\tilde\beta_i + \mu^2_{\beta_i}\tilde\beta_i^2$$

> all decoupled fluctuations!

> but masses blow up at turning points $\qquad m^2_{\bar\phi} = 2\,\rho'^2 + 2\frac{\kappa^2\,\omega^2}{\rho'^2}$

> superficial UV divergence $\sim \int d\tau d\sigma\,\sqrt{-g}\,R^{(2)}$?

⟶ Up to now conformal gauge preferred

[Drukker Gross Tseytlin 00]
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$\longrightarrow$ Up to now conformal gauge preferred

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HOWEVER

> Equivalence of semiclassical (1-loop) partition fcs. for Nambu and Polyakov action.

[Fradkin, Tseytlin 82]

> $\sqrt{-g}\, R^{(2)}$: total derivative, sensitive to topology of induced world-sheet (cylinder)

▸ **A posteriori** **proof** ***conformal gauge = static gauge***

✓ *UV finiteness of static gauge action*

✓ *Static gauge results reproduce conformal gauge ones*

- **A posteriori** **proof *conformal gauge = static gauge***

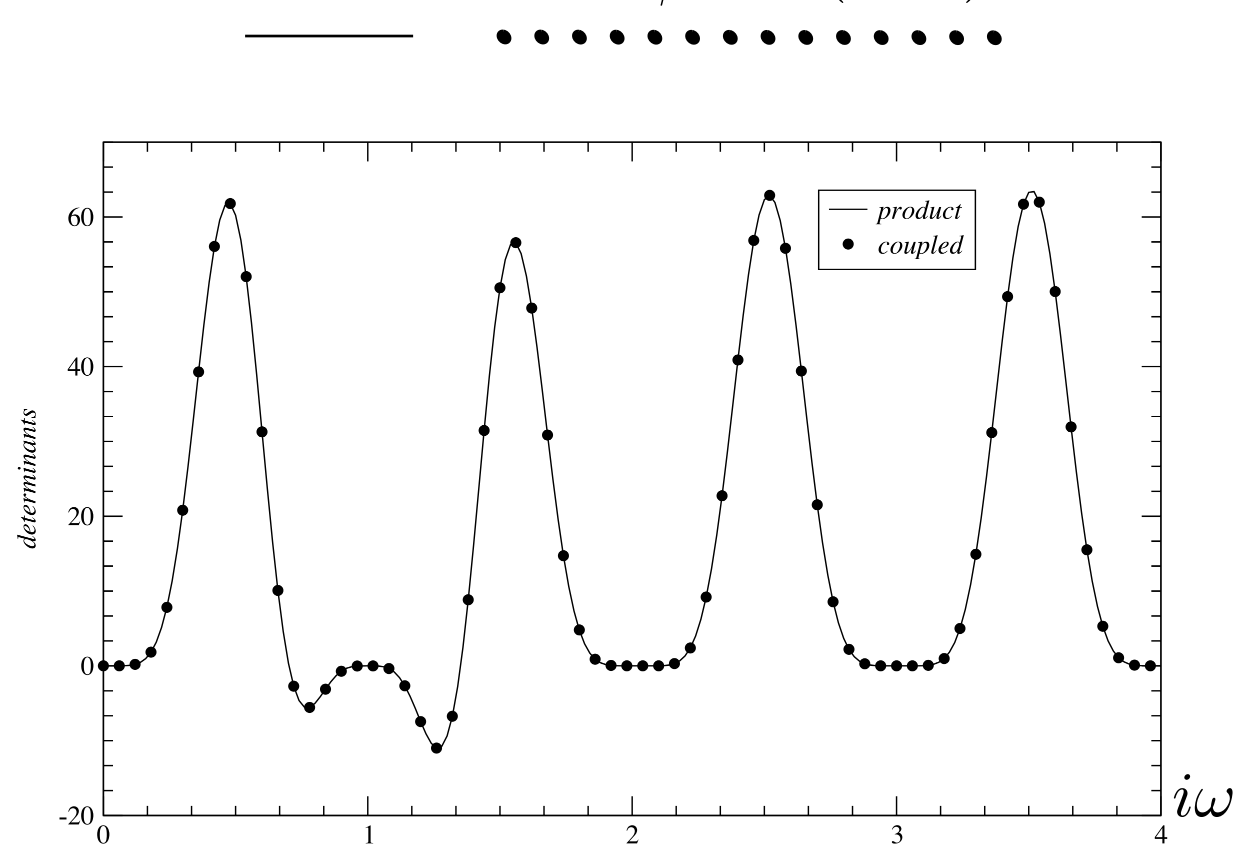
  ✓ *UV finiteness of static gauge action*

  ✓ *Static gauge results reproduce conformal gauge ones*

  ✓ *Equivalence (numerical) of determinants!*

$$\Gamma_1^{\rm CG} = -\frac{\mathcal{T}}{4\pi}\int_{\mathbb{R}} d\omega \frac{\det^8 \mathcal{O}_\psi}{\det^2 \mathcal{O}_\beta \, \det Q \; \det^3(-\partial^2)} \qquad \Gamma_1^{\rm SG} = -\frac{\mathcal{T}}{4\pi}\int_{\mathbb{R}} d\omega \frac{\det^8 \mathcal{O}_\psi}{\det^2 \mathcal{O}_\beta \, \det \mathcal{O}_\phi \; \det^5(-\partial^2)}$$

$$\det Q = \det \mathcal{O}_\phi \; \det^2(-\partial^2)$$



How one could proceed... (non "exactly")

- Non trivial masses $m^2(\sigma) \sim \rho'^2 = \kappa^2\, \mathrm{cn}^2\left[\frac{\kappa\,\sigma}{\epsilon}, -\epsilon^2\right]$

- Expanding, leading order is sigma independent: ok! $\rho'^2 \to \kappa_0^2$ $\qquad k_0 = \frac{1}{\pi}\ln[16\,\epsilon^2]$

$$\Gamma^{(0)} = \mathcal{T}\int\frac{d\omega}{2\pi}\ln\frac{\det(-\partial_1^2+\omega^2+\kappa_0^2)^8}{\det(-\partial_1^2+\omega^2+4\kappa_0^2)\,\det(-\partial_1^2+\omega^2+2\kappa_0^2)^2\,\det(-\partial_1^2+\omega^2)^5}$$

> Leading contribution (constant mass relativistic fields on the cylinder!)

$$E^{(0)} = \frac{1}{2\,\kappa}\sum_{n=-\infty}^{\infty}\left[2\sqrt{n^2+2\kappa_0^2}+\sqrt{n^2+4\kappa_0^2}+5\sqrt{n^2}-8\sqrt{n^2+\kappa_0^2}\right]$$ **[Frolov, Tseytlin 02]**

Euler-MacLaurin →

$$E_1^{(0)} = \frac{\Gamma_1^{(0)}}{\kappa\,\mathcal{T}} = \frac{1}{\kappa}\left[-3\ln 2\,\kappa_0^2 - \frac{5}{12}+\mathcal{O}(e^{-2\pi\kappa_0})\right], \qquad \kappa_0\to\infty$$

✓ 1-loop correction to cusp anomaly $\ln\epsilon^2 \sim \ln\mathcal{S}$

[Frolov, Tseytlin 02]

[Schaefer-Nameki, Zamaklar 05]

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- **BUT** further expansion breaks down!!

$$\rho'^2 = \kappa_0^2 - \frac{1}{\epsilon^2}\kappa_0\left[\pi\kappa_0\,\cosh(2\kappa_0\sigma)-2\right]+\ldots \sim \frac{1}{\epsilon^2}e^{\frac{2\sigma}{\pi}\ln 16\epsilon^2} \sim \left(\frac{1}{\epsilon}\right)^0$$ at turning points

The **exact** way

- Eigenvalue fluctuation equation, eg. two fluctuations β_i $(m^2_{\beta_i} = 2\rho'^2)$

$$\Big\{ -\partial_\sigma^2 + \omega^2 + 2\rho'^2 \Big\}\, \beta_i(x) = \lambda\, \beta_i(x)$$

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$$\Big\downarrow \quad k^2 = \frac{\epsilon^2}{1+\epsilon^2} \qquad x = \frac{2\mathbb{K}}{\pi}\,\sigma \quad \longrightarrow \quad \beta_i(x+4\mathbb{K}) = \beta_i(x)$$

$$\boxed{\Big\{ -\partial^2_x + 2k^2\,\mathrm{sn}^2[x+\mathbb{K},k^2] + \Omega^2 \Big\}\,\beta_i(x) = \lambda\,\beta_i(x)}$$

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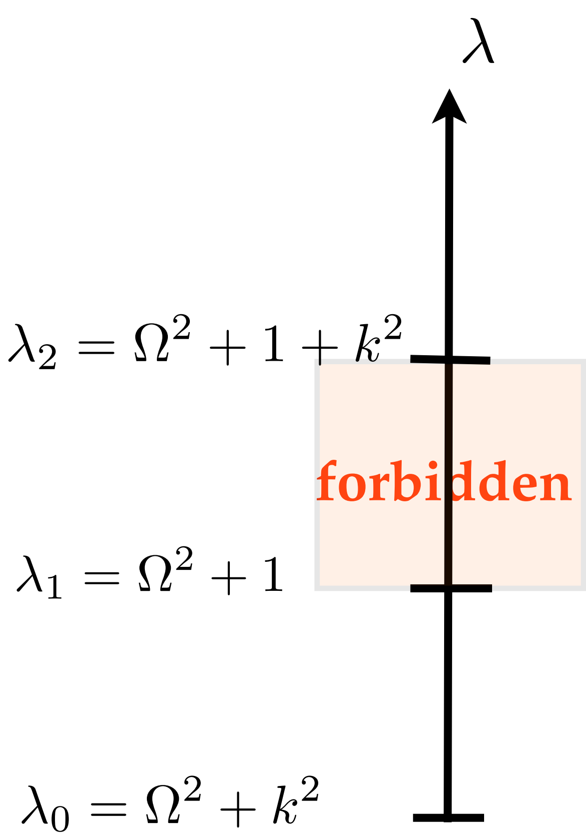
Lamé equation

with periodic b.c.

Case j=1 of the Lamé equation in Jacobian form

$$\Big\{ -\partial_x^2 + 2\,j(j+1)\,k^2\,\mathrm{sn}^2[x, k^2] - h \Big\}\, \Psi = 0$$

The spectral problem of Lamé potential



- Band structure determined by properties of Floquet exponent (quasi-momentum)

$$\beta_\pm(x + 4\mathbb{K}) = e^{\pm i\,F}\,\beta_\pm(x)$$

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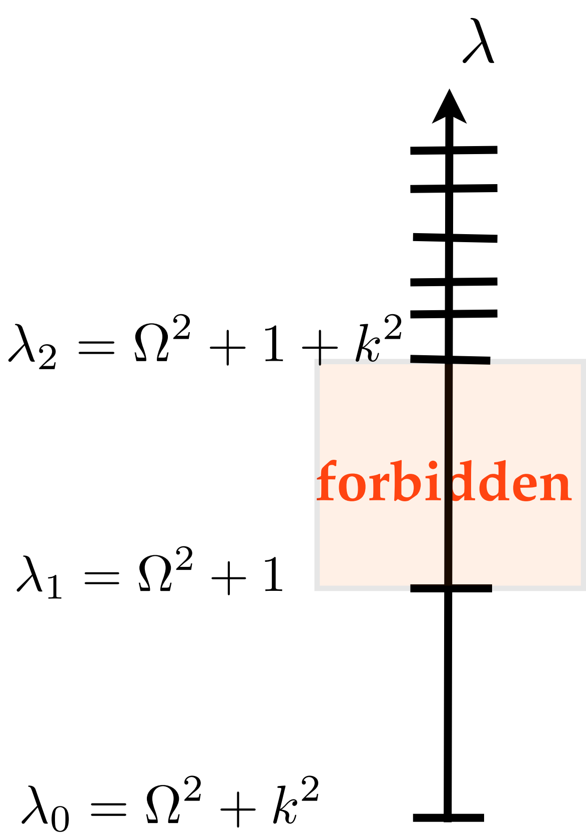

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- Periodic boundary conditions: spectrum discrete

$$\lambda_n = \frac{2-k^2}{3} - \mathcal{P}(i\,y_n)$$

$$F(i\,y_n) = 2\mathbb{K}\,i\,\zeta(i\,y_n) + 2\,y_n\,\zeta(\mathbb{K}) = 2\,\pi\,n \qquad n = 1, 2, ...$$

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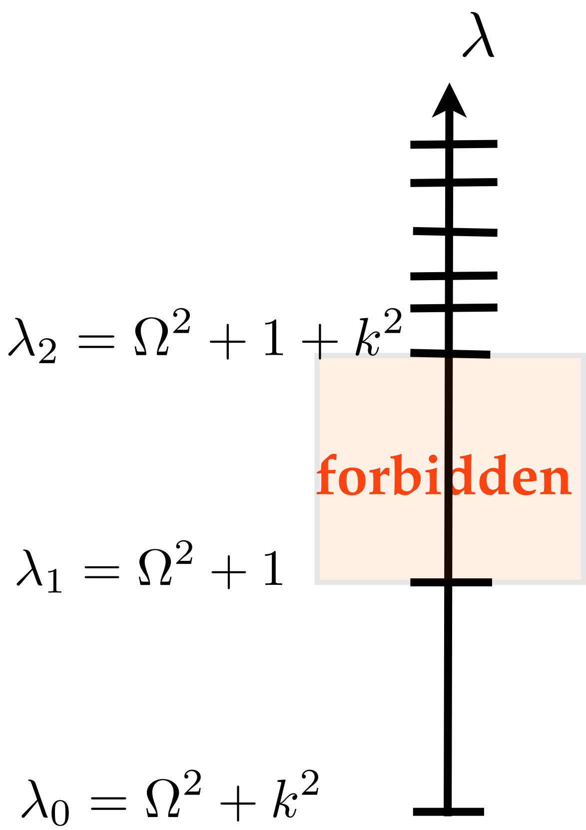

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The spectral way to functional determinants is feasible
[only need a finite subset of eigenvalues: the 3 band edges!]

but there is a powerful short-cut

Gel'fand-Yaglom Theorem (1960)

- Consider $K_g(x) = -\partial_x^2 + g\, V(x)$ for $x \in [0, L]$ and $g \in (0, 1)$

  $K_g(x)\,\phi(x) = \lambda\,\phi(x)$ with <u>Dirichlet bc</u> $\phi(0) = \phi(L) = 0$

- To compute the determinant, solve the <u>initial value problem</u>

$$K_g(x)\,\bar{\phi}(x) = 0 \qquad \bar{\phi}(0) = 0 \qquad \bar{\phi}'(0) = 1$$

$$\Rightarrow \qquad \det \frac{K_g}{K_0} = \frac{\bar{\phi}(L)}{L}$$

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$$\Rightarrow \quad \det \frac{K_g}{K_0} = \frac{\bar{\phi}(L)}{L}$$

Example: Helmoltz operator $[-\partial_x^2 + m^2]$ with Dirichlet b.c.

> Dirichlet spectrum $$\frac{\det[-\partial_x^2 + m^2]}{\det[-\partial_x^2]} = \prod_{n=1}^{\infty} \frac{m^2 + (\frac{n\,\pi}{L})^2}{(\frac{n\,\pi}{L})^2} = \frac{\sinh m\, L}{m\, L}$$

> Gel'fand-Yaglom $$\frac{\det[-\partial_x^2 + m^2]}{\det[-\partial_x^2]} = \frac{\phi(x)}{\phi_0(x)} = \frac{\sinh(m\, L)}{mL}$$

$$\phi(x) = \sinh(m\, x) \qquad \phi_0(x) = x$$

GY at work for spinning string

- Gel'fand-Yaglom for periodic b.c. $x \in [0, P]$

given $y_1(0) = 1 \quad y_1'(0) = 0$

$y_2(0) = 0 \quad y_2'(0) = 1$

$\Rightarrow \quad \det \mathcal{O} = y_1(P) + y_2'(P) - 2$

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$y_2(0) = 0 \quad y_2'(0) = 1$ ➡ $\det \mathcal{O} = y_1(P) + y_2'(P) - 2$

- Solutions of the associated homogeneous equation **[Hermite 1872]**

$$\beta_\pm(x) = \frac{H(x \pm \alpha)}{\Theta(x)}\, e^{\mp Z(\alpha) x} \qquad Z(u) = \frac{\pi}{2\mathbb{K}} \frac{\theta_4'\left(\frac{\pi u}{2\mathbb{K}}, q\right)}{\theta_4'\left(\frac{\pi u}{2\mathbb{K}}, q\right)} \qquad H(u) = \theta_1\left(\frac{\pi u}{2\mathbb{K}}, q\right)$$

$$\mathrm{sn}(\alpha, k^2) = \sqrt{1 + \frac{1}{k^2}\left(1 + \frac{\pi^2\,\omega^2}{4\,\mathbb{K}^2(k^2)}\right)} \qquad \Theta(u) = \theta_4\left(\frac{\pi u}{2\mathbb{K}}, q\right)$$

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- Gel'fand-Yaglom for <u>periodic b.c.</u> $x \in [0, P]$

given $y_1(0) = 1$ $\quad y_1'(0) = 0$

$y_2(0) = 0$ $\quad y_2'(0) = 1$ $\quad \Longrightarrow \quad \det \mathcal{O} = y_1(P) + y_2'(P) - 2$

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$$\mathrm{sn}(\alpha, k^2) \quad = \quad \sqrt{1 + \frac{1}{k^2}\left(1 + \frac{\pi^2\,\omega^2}{4\,\mathbb{K}^2(k^2)}\right)} \qquad \Theta(u) = \theta_4\left(\frac{\pi u}{2\mathbb{K}}, q\right)$$

- The <u>result</u>

Using GY for $P = 4\mathbb{K}$ and exploiting $\beta_\pm(x + 4\mathbb{K}) = e^{\mp 4 Z(\alpha)\mathbb{K}} \beta_\pm(x)$

$$\det \mathcal{O}_\beta = 4 \sinh^2 2\mathbb{K}\, Z(\alpha)$$

The exact way II: ubiquitous Lamé !

- Bosonic mode $\bar{\phi}$, $m^2_{\bar{\phi}} = 2\rho'^2 + \dfrac{2\kappa^2\omega^2}{\rho'^2}$

$$\left\{ -\partial_x^2 + 2\,k^2\,\mathrm{sn}^2[x,k^2] + \frac{2}{\mathrm{sn}^2[x,k^2]} + \Omega^2 \right\} \bar{\phi}(x) = 0$$

↓ modular transformations 2

$$\boxed{\left\{ -\partial_x^2 + 2\,\tilde{k}^2\,\mathrm{sn}^2\left[x,\tilde{k}^2\right] + \tilde{\Omega}^2 \right\} \bar{\phi}(x) = 0}$$

Lamé equation 2

- Fermions $\hat{\mu}^2_{F_\pm} = \pm\rho'' + \rho'^2$

$$\left\{ -\partial_x^2 + k^2\,\mathrm{sn}^2[x,k^2] \pm\, k^2\,\mathrm{cn}^2[x,k^2]\,\mathrm{dn}^2[x,k^2] + \Omega^2 \right\} \bar{\psi}(x) = 0$$

↓ modular transformation 3

$$\boxed{\left\{ -\partial_x^2 + \hat{k}^2\,\mathrm{sn}^2[x,\hat{k}^2] + \hat{\Omega}^2 \right\} \bar{\psi}(x) = 0}$$

Lamé equation 3

The exact way III: Exact expression for 1-loop semiclassical energy

Given the determinants

$$\det\mathcal{O}_\beta = \sinh^2[2\,\mathbb{K}(k^2)\,Z(\alpha)] \qquad \mathrm{sn}(\alpha,k^2) = \sqrt{1+\frac{1}{k^2}\Big(1+\frac{\pi^2\,\omega^2}{4\,\mathbb{K}^2(k^2)}\Big)}$$

$$\det\mathcal{O}_\phi = \sinh^2\Big[\mathbb{K}\big(\frac{4k}{(1+k)^2}\big)\,Z(\tilde\alpha)\Big] \qquad \mathrm{sn}\Big(\tilde\alpha,\frac{4k}{(1+k)^2}\Big) = \sqrt{\frac{(1+k)^2}{8k}\Big[2+\frac{\pi^2\,\omega^2}{\mathbb{K}^2(\frac{4k}{(1+k)^2})}\Big]}$$

$$\det\mathcal{O}_\psi = \cosh^2\Big[\mathbb{K}\big(\frac{4k}{(1+k)^2}\big)\,Z(\hat\alpha)\Big] \qquad \mathrm{sn}\Big(\hat\alpha,\frac{4k}{(1+k)^2}\Big) = \sqrt{\frac{(1+k)^2}{4k}\Big[1+\frac{\pi^2\,\omega^2}{\mathbb{K}^2(\frac{4k}{(1+k)^2})}\Big]}$$

the one-loop effective action reads

$$\Gamma_1 = -\frac{\mathcal{T}}{4\pi}\int_{\mathbb{R}} d\omega\,\ln\frac{\det^8\mathcal{O}_\psi}{\det^2\mathcal{O}_\beta\;\det\mathcal{O}_\phi\;\det^5(-\partial^2)}$$

5 trivial fluctuations in S^5

from which the one-loop energy

$$E_1 = \frac{\Gamma_1}{\kappa\mathcal{T}}\,, \qquad \mathcal{T}\equiv\int d\tau \to \infty$$

UV-finiteness

The behavior of the integrand for $\omega \to \infty$

$$\ln \det \mathcal{O}_i = r_0\, \omega + \frac{r_{1,i}}{\omega} + \mathcal{O}(\omega^{-3}),$$

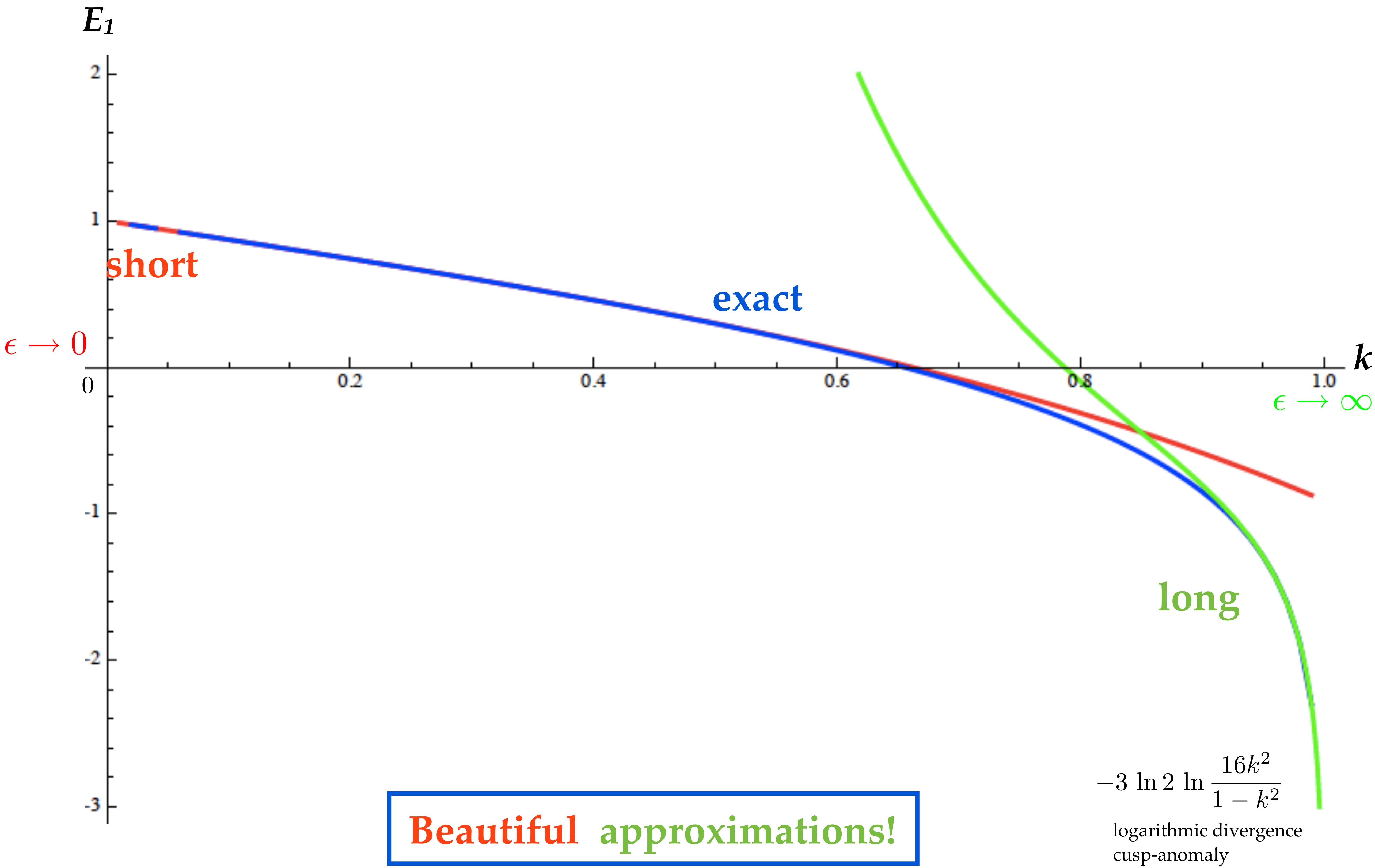
is determined by the fluctuations potentials $V_{\beta,\,\phi,\,\zeta_i,\,\psi}$

$$r_0 = 2\pi \qquad r_{1,i} = \pi\,\langle V_i \rangle \qquad \text{e.g. } \langle V_\beta \rangle = \frac{1}{4\mathbb{K}} \int_0^{4\mathbb{K}} \mathrm{sn}^2(x|k^2)$$

Collecting altogether

$$\ln \frac{\det^8 \mathcal{O}_\psi}{\det^2 \mathcal{O}_\beta\ \det \mathcal{O}_\phi\ \det^5 \zeta} \sim \frac{2\mathbb{K}}{\pi}(\mathbb{K} - \mathbb{E}) \Big[8 \times 2 - 2 \times 4 - 1 \times 8 - 5 \times 0\Big] = 0$$

1-loop energy: exact vs. expanded



Beautiful approximations!

Long strings - Large Spin Expansion

- Leading $\epsilon \to \infty$ behavior. Constant potential fluctuations $\rho' \approx \kappa_0$, cusp anomaly.

- Expanding further $$\Gamma_1^{\rm NLO} = \frac{\mathcal{T}}{\pi}\,\kappa_0\,(\pi + 6\,\ln 2), \qquad \kappa_0 \to \infty$$

recover a first correction missing in previous analysis! (turning point contribution)

[Gromov unpublished 09]

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- Going to many subleading orders in $1/\epsilon^6$, **namely in** $1/\mathcal{S}^3$

$$\mathcal{E}_1 = \frac{\kappa_0}{\kappa}\frac{1}{\pi}\left[c_{01}\kappa_0 + c_{00} + \frac{d_{01}}{\kappa_0} + \frac{1}{\epsilon^2}\left(c_{10} + \frac{d_{11}}{\kappa_0}\right) + \frac{1}{\epsilon^4}\left(c_{21}\kappa_0 + c_{20} + \frac{d_{21}}{\kappa_0}\right) + \frac{1}{\epsilon^6}\left(c_{31}\kappa_0 + c_{30} + \frac{d_{31}}{\kappa_0}\right) + \ldots\right]$$

$$\begin{aligned}
c_{01} &= -3\pi\log 2, & c_{00} &= \pi + 6\log 2, & d_{01} &= -\frac{5\,\pi}{12},\\
c_{11} &= 0, & c_{10} &= -3\log 2, & d_{11} &= \frac{1}{2} + \frac{3\,\ln 2}{\pi}\\
c_{21} &= -\frac{\pi^2}{32} - \frac{3}{32}\pi\log 2, & c_{20} &= \frac{\pi}{16} + \frac{39\log 2}{32}, & d_{21} &= -\frac{13}{64} - \frac{63\log 2}{32\pi},\\
c_{31} &= \frac{\pi^2}{32} + \frac{3}{32}\pi\log 2, & c_{30} &= -\frac{3\pi}{32} - \frac{13\log 2}{16}, & d_{30} &= \frac{29}{192} + \frac{85\log 2}{64\pi}
\end{aligned}$$

Expansion compatible with ***reciprocity*** **?**

At weak coupling, ***reciprocity*** is an observed <u>regularity</u> in the large spin expansion of the anomalous dimension for *twist operators.*

Reciprocity at weak coupling

reviewed in
[Beccaria, Forini, Macorini, 10]

- Operators $\mathcal{O} = \mathrm{Tr}\{D^{k_1}X \ldots D^{k_J}X\}$ $\quad k_1 + \ldots + k_J = S$

Rephrase the large S expansion of γ in terms of another function f

$$\gamma = \mathrm{f}(S + \tfrac{1}{2}\gamma - \tfrac{1}{2}\beta)$$ ☆

☆ In QCD

the ***evidence*** is that f has a (large S) ***parity invariant*** $C \to -C$ expansion

$$\mathrm{f}(S) = \sum_n \frac{a_n(\ln C)}{C^{2n}}$$

[Basso, Korchemsky 06]
[Dokshitzer, Marchesini 06]

$C^2 = (S + J\,\ell)(S + J\,\ell - 1)$ ***Casimir of*** SL(2,R) $\subset$ SO(4,2)

J : twist $\quad \ell$:

φ	λ	A
$\frac{1}{2}$	1	$\frac{3}{2}$

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- In Mellin space, ***parity invariance*** becomes

$$\mathrm{F}(x) = -x\,\mathrm{F}\left(\frac{1}{x}\right) \qquad \text{where} \qquad \mathrm{f}(x) = \int_0^1 dx\, x^{S-1}\,\mathrm{F}(x)$$

or a ***generalized*** (*Gribov-Lipatov*) ***reciprocity***.

[Gribov, Lipatov, 72]

Evidence at weak coupling

✔ All twist-2 anomalous dimensions in QCD (3 loops) [Basso, Korchemsky 06]

✔ Twist 2-3 in various sectors of $\mathcal{N}$=4 SYM also with wrapping [Beccaria, Marchesini, Dokshitzer 07] [Beccaria 07] [Beccaria Forini 08]

$\mathcal{O}$	# loops	wrapping	reciprocity
$\langle\varphi\varphi\rangle$, $\langle\psi\psi\rangle$, $\langle AA\rangle$	5	yes	$\surd$
$\langle\varphi\varphi\varphi\rangle$	5	yes	$\surd$
$\langle\psi\psi\psi\rangle$	5	yes	$\surd$
$\langle AAA\rangle$	4	no	$\surd$ (ABA)

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- Reciprocity **has been even <u>assumed</u>** to simplify multiloop calculations

Ex.1 Twist three at 5 loops $\quad \mathrm{Tr}\left(\mathcal{D}^{s_1} Z\, \mathcal{D}^{s_2}\, Z\, \mathcal{D}^{s_3} Z\right) \quad$ with $\quad S = s_1 + s_2 + s_3$

Reciprocity-respecting ansatz for the anomalous dimension. [Beccaria,Forini, Lukowski, Zieme 09]

✓ Verified with Y-system! [Gromov, Kazakov, Vieira 09]

✓ Verified field-theoretically! [Fiamberti, Santambrogio, Sieg 09]

Ex.2 Twist two at 5 loops $\quad \mathrm{Tr}\left(\mathcal{D}^{s_1} Z\, \mathcal{D}^{s_2} Z\right) \quad$ with $\quad S = s_1 + s_2$

[Rej, Lukowski, Velizhanin 09]

Reciprocity at strong coupling

- From "anomalous dimension"

$$\Delta_0(\mathcal{S}) = \mathcal{E}_0 - \mathcal{S} \qquad\qquad \Delta_1(\mathcal{S}) = \mathcal{E}_1$$

define $\mathcal{F}$ via

$$\Delta(\mathcal{S}) = \mathcal{F}\Big(\mathcal{S} + \frac{1}{2}\Delta(\mathcal{S})\Big)$$

$$\Delta(\mathcal{S}) = \Delta_0(\mathcal{S}) + \frac{1}{\sqrt{\lambda}}\Delta_1(\mathcal{S}) + \cdots \longrightarrow \mathcal{F}(\mathcal{S}) = \mathcal{F}_0(\mathcal{S}) + \frac{1}{\sqrt{\lambda}}\mathcal{F}_1(\mathcal{S}) + \cdots$$

and expand at large $\mathcal{S}$.

- Re-express in terms of the "semiclassical" Casimir $\mathcal{C} \equiv \mathcal{S}$

$$C^2 = S(S+1) \longrightarrow \times\frac{1}{(\sqrt{\lambda})^2} \longrightarrow \mathcal{C}^2 = \mathcal{S}\Big(\mathcal{S} + \frac{1}{\sqrt{\lambda}}\Big)$$

- Coefficients of odd terms under $\mathcal{S} \to -\mathcal{S}$ vanish!

$$\begin{aligned} c_{10} &= \frac{1}{\pi}\,c_{01}, \qquad d_{11} = \frac{1}{2\pi}\,c_{00}, \qquad c_{30} = -c_{20} - \frac{1}{6\pi}c_{01} + \frac{1}{\pi}\,c_{21} \\ c_{31} &= -c_{21}, \qquad d_{31} = -d_{21} + \frac{1}{4\pi^2}\,c_{01} - \frac{1}{12\pi}c_{00} + \frac{1}{2\pi}\,c_{20}. \end{aligned}$$

✓ Reciprocity holds up to $1/\mathcal{S}^3$

Short strings

- Realized sending $\epsilon \to 0, \quad k \to 0$

$$\det\, \mathcal{O}_{f=\beta,\phi,\psi} = D_f^{(0)}(\omega) + \epsilon^2\, D_f^{(1)}(\omega) + \epsilon^4\, D_f^{(2)}(\omega) + \cdots\,,$$

- Isolating lowest eigenvalues

$$E_1 = 1 - \frac{1}{4\pi\kappa} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \ln \frac{(\det'\, \mathcal{O}_\psi)^8}{(\det'\, \mathcal{O}_\beta)^2 \det'\, \mathcal{O}_\phi}$$

- The 1-loop correction in the short string limit reads

$$\begin{aligned} E_1^{\rm an} &= \sqrt{2\,\mathcal{S}}\left(\frac{3}{2} - 4\ln\, 2\right) + \frac{1}{\sqrt{2}}\left(\frac{-46 + 48\ln\, 2 + 24\,\zeta(3)}{16}\right)\mathcal{S}^{3/2} \\ &\quad + \frac{1}{\sqrt{2}}\left(\frac{1378 - 1008\ln\, 2 - 240\,\zeta(3) - 480\,\zeta(5)}{256}\right)\mathcal{S}^{5/2} + \mathcal{O}(\mathcal{S}^{7/2}) \\ E_1^{\rm nan} &= 1 + \mathcal{O}(\mathcal{S}) \end{aligned}$$

- **<u>Disagreement</u>** with semiclassical evaluation based on algebraic curve approach.

[Gromov, unpublished]

Work in progress: QFT vs geometry

[Beccaria, Dunne, Forini, Pawellek,Tseytlin]
+ [Gromov]

- QFT: fluctuations governed by the single-gap operators

  → *Riemann surface and elliptic curve interpretation* [Belokos, Bobenko, Enolskii, Its, Matseev, 94]

- Algebraic curve approach to integrability

  > Classical string sols map to Riemann surfaces with several sheets and cuts.

  → Classical energy is a contour integral. [Kazakov, Marshakov, Minahan, Zarembo 04] [Beisert, Kazakov, Sakai, Zarembo 05]

  > Semiclassical quantization: pinching the surface by adding *extra cuts* [Gromov, Vieira 07]

$$\delta E_{1-\text{loop}} = \frac{1}{2} \sum_{n,ij} (-1)^{F_{ij}} \, \Omega_n^{ij}$$

i,j: 8+8 bos. and ferm. polarizations

$F_{ij} = \pm 1$

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1. Non trivial dictionary to be constructed!

> From alg.curve quasi-momentum and eigenfrequencies (x: spectral paramer)

$$\frac{dp}{dx}\frac{dx}{d\omega} \equiv \frac{dp}{d\omega} \sim \frac{\omega^2 + f(k^2)}{\sqrt{(\omega_1^2+\omega^2)\,(\omega_2^2+\omega^2)\,(\omega_3^2+\omega^2)}}$$

our single gap problem

2. Interesting also to explain *disagreements* between the two approaches

Concluding remarks & perspectives

✓ **<u>Exact</u>** starting point for 1-loop corrections to the energy of folded string: fluctuations with ubiquitous, diagonalizable, Lamé operators

- Generalization to (S,J) solution. Still finite gap expected!
  <u>caveat</u>: fluctuations <u>coupled</u> even in static gauge

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✓ **<u>AdS/CFT</u>**: short string limit (<u>not confirming</u> previous results), QCD-like properties for large spin structure

✓ **<u>Integrability</u>**: "inherited" from classical solution, "redescovered" with the integrable Lamé equation.

- Generalization to (S,J) solution. Still finite gap expected!
  <u>caveat</u>: fluctuations <u>coupled</u> even in static gauge

- (Detailed) comparison with algebraic curve approach

- Classify integrable matrix differential operators corresponding to sigma model classical solutions.