Spacelike minimal surfaces in $AdS \times S$

In collaboration with
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Outline

- **Introduction**
- **Pohlmeyer reduction in general**
  (tool to construct the string solution)
- **Pohlmeyer reduction in $AdS_3 \times S^3$**
- **The AdS_3 projection**
  (description of the new string solutions)
- **The S^3 projection**
- **Regularized area**
- **Conclusions**

Motivation-Introduction

- In 0705.0303 Alday-Maldacena conjectured that planar gluon scattering amplitudes at strong coupling = area of classical string configuration in $AdS \times \mathrm{point}$ with light-like boundaries.
- After that extension to AdS_4, AdS_5, 8-gluons, n-gluons.
- Here, we ask the question what happens when we have $AdS_5 \times S^5$ strings with light-like boundaries.

We want to study spacelike minimal surfaces in $AdS \times S$. There are two possibilities.

- separately minimal in AdS and minimal in S
  $\mathrm{Virasoro}_{AdS} = \mathrm{Virasoro}_S = 0$
  example was given by Alday-Maldacena
  $Y = \frac{1}{\sqrt{2}}(\cosh\tau, \cosh\sigma, \sinh\sigma, \sinh\tau)$
  sphere part can be a point or everything but a point or twice everything but a point, etc (instanton solution, stereographic projection)

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  sphere part can be a point or everything but a point or twice everything but a point, etc (instanton solution, stereographic projection)
- minimal only in total $AdS \times S$ but not in separately
  $\mathrm{Virasoro}_{AdS} + \mathrm{Virasoro}_S = 0, \quad \mathrm{Virasoro}_{AdS} \neq \mathrm{Virasoro}_S$
  We have two possibilities
  - AdS spacelike, S spacelike
  - AdS timelike, S spacelike

Pohlmeyer reduction (1976)

We would like to solve eom+Virasoro for the string sigma model. Many different techniques have been developed (depending on the problem we want to solve) including

- ansatz
- dressing method (for example in the case of giant magnons)
- Pohlmeyer reduction
- ...

We use the Pohlmeyer reduction method. One can view the Pohlmeyer reduction as a sophisticated gauge choice where we are left with a model that only involves physical degrees of freedom. The reduced model inherits integrable structures of the original sigma model.

Let us see how the Pohlmeyer reduction works in the simpler case of pure AdS_3 and then quote the result of the $AdS \times S$ case we are interested in.

AdS_3 example

$$\text{eom}: \quad \partial\bar{\partial}Y = (\partial Y \cdot \bar{\partial}Y)Y$$

$$\text{Vir}: \quad (\partial Y)^2 = (\bar{\partial}Y)^2 = 0$$

$$\text{length}: \quad Y^2 = -1$$

$\rightleftarrows$

sinh-Gordon

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$\sinh$-*Gordon should contain all information we need in boundary behavior and location of the poles.*

Some more examples

$$
\begin{aligned}
S^2 \text{ strings} &\longleftrightarrow \sin\text{Gordon} \\
S^3 \text{ strings} &\longleftrightarrow \text{complex}\ \ \sin\text{Gordon} \\
AdS_5 \text{ strings} &\longleftrightarrow \text{generalized}\ \ \sinh\text{Gordon} \\
CP^3 \text{ strings} &\longleftrightarrow \text{known} \\
AdS_5 \times S^5 \text{ strings} &\longleftrightarrow \text{system of generalized}\ \ \sin(\text{h})\ \text{Gordon}
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strings in $AdS_5 \times S^5$		generalized $\sin(\text{h})$-Gordon
ansatz		Hitchin equations
dressing method	$\rightleftarrows$	ansatz
Bäcklund transformation		Bäcklund transformation
…		…

AdS_3 projection

We focus on strings in $AdS_3 \times S^3$. It turns out that the solution depends on four real parameters, two for the sphere projection and two for the AdS_3. Let us call these parameters

$$\theta, \theta_s, \rho, \rho_s,$$

where the subscript s means sphere.
The AdS_3 projection is

$$Y = (\sin\theta\cosh\eta,\ \cos\theta\cosh\xi,\ \cos\theta\sinh\xi,\ \sin\theta\sinh\eta).$$

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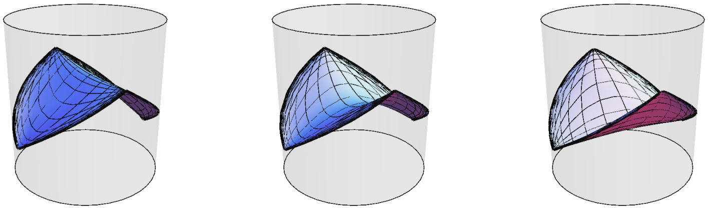
$$Y = (\sin\theta \cosh\eta,\ \cos\theta \cosh\xi,\ \cos\theta \sinh\xi,\ \sin\theta \sinh\eta).$$

• η, ξ are linear combinations of the worldsheet coordinates σ, τ and they depend on ρ, θ.

• It depends on two parameters θ, ρ.

• the meaning of the parameter ρ will be discussed later.

• It is the intersection of the AdS hyperboloid and the surface $Y_0^2 - Y_1^2 = \cos^2\theta$.



$\theta \to 0$ $\theta = \pi/4$ $\theta \to \pi/2$

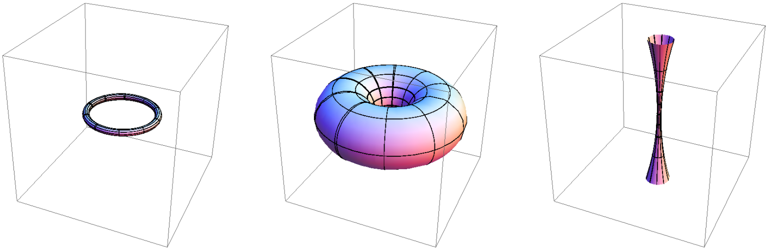
Figure: *Three different AdS_3 solutions. The plot in the middle corresponds to the solution of Alday-Maldacena.*

- The AdS_3 surfaces has constant mean curvature.
- The shape of the surface only depends on one parameter, θ.
- It is not minimal, but it becomes minimal when $\theta = \pi/4$ and then we recover the Alday-Maldacena surface.
- They all have the same lightlike boundaries.

S^3 projection

Similarly, the S^3 projection is

$$X = (\sin\theta_s \cos\eta_s,\ \cos\theta_s \cos\xi_s,\ \cos\theta_s \sin\xi_s,\ \sin\theta_s \sin\eta_s).$$



$\theta_s \to 0$ $\theta_s = \pi/4$ $\theta_s \to \pi/2$

Figure: *Three plots showing a stereographic projection of the above sphere solutions with different values of the parameter θ_s. The left plot shows that the case $\theta_s \to 0$ maps to a circle. The right plot is the solution for $\theta_s \to \pi/2$, which is a similar degenerate torus, but now projected to an infinite line in the stereographic projection.*

Explanation of the meaning of ρ, ρ_s

They are parameters of the inner geometry.
Consider the toy model

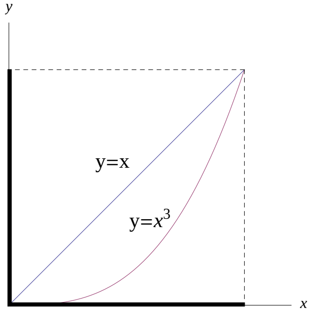


Figure: Two different curves with the same projections.

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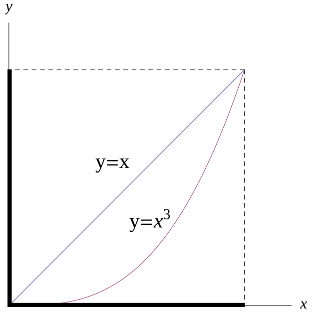

Figure: Two different curves with the same projections.

- So, ρ, ρ_s control the relative orientation of the AdS_3 and S^3 projections.
- They count how many cm I move in S^3, if I move 1 cm in AdS_3.
- The total induded metric is conformal and equal to $(1 + \rho^2 + \rho_s^2) I_{2\times 2}$.

Regularized action

There are several methods to regularize, including dimensional regularization and using a cutoff r_c.
For our solution we have found that the regularized area is

$$S_{reg} = \frac{\sqrt{\lambda}}{2\pi} \frac{(1+\rho^2+\rho_s^2)\sin 2\theta}{\rho\sqrt{1+\rho^2}} I(r_c) \ ,$$

$$I(r_c) = \frac{1}{4}\left(\log\frac{r_c^2\cos^2\theta}{-4\pi^2 s}\right)^2 + \frac{1}{4}\left(\log\frac{r_c^2\sin^2\theta}{-4\pi^2 t}\right)^2 - \frac{1}{4}\left(\log\frac{s}{t}\right)^2 + \text{const.}$$

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- s, t are the Mandelstam variables
- $I(r_c)$ has the same (s,t)-dependent part as the BDS formula with a suitable position dependent cutoff
- in general $\text{prefactor} > 1$, for $\rho \to \infty, \theta = \pi/4$ we get $\text{prefactor} = 1$

Spacelike strings with timelike AdS_3

By analytically continuing some of the parameters of our spacelike in total $AdS_3 \times S^3$ solution (with spacelike AdS_3 projection) we can get a new family of spacelike in total string solutions (with timelike AdS_3 projection).

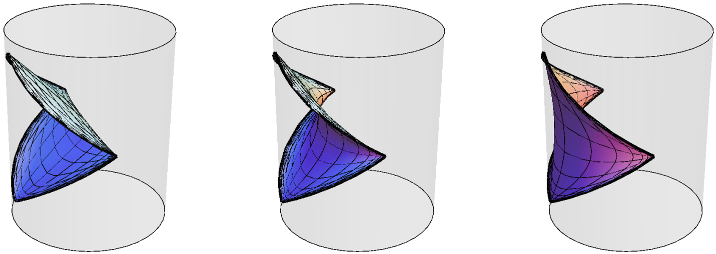

Figure: *The AdS_3 projection of different solutions. There are more solutions that are not presented here. A more detailed study and classification of solutions with lightlike boundaries is in progress.*

Timelike in total strings

A second by-product of our construction is a set of new timelike surfaces in $AdS_3 \times S^3$, that are not minimally in AdS_3 and S^3 separately. We just plot some representatives.

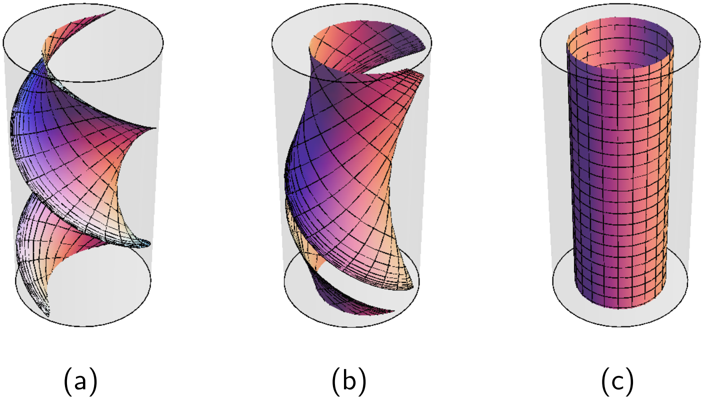

Figure: *AdS projection of time-like surfaces in $AdS \times S$.*

Summary-Conclusions

- We have constructed a four parameter family of string solutions in $AdS_5 \times S^5$ whose boundary approaches the light-like tetragon and are a generalization of the solution of Alday-Maldacena. These minimal surfaces are space-like and flat. Their projections on each of the AdS_5 and S^5 have constant mean curvature. As the surface approaches the boundary of AdS_5 it wraps a torus inside S^5 an infinite number of times. The solutions therefore satisfy Neumann boundary conditions on S^5.
- We have demonstrated the use of a general and powerful method (due to Pohlmeyer) that reduces the string system to a system with only physical degrees of freedom.
- Pohlmeyer reduction has some classification power (work in progress)
- Up to a prefactor our area is the same as the one of Alday-Maldacena with a position dependent cutoff.
- Are our solutions related to scattering amplitudes?
- Wilson loops?