Defects, dualities and gauging in string theory via gerbes

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(based, in part, on joint work with Gawędzki & Waldorf, with Runkel)

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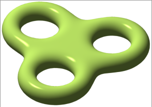
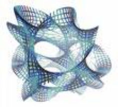
Part I

A 2-category for the $2D$ σ-model

The gerbe for the monophase world-sheet

Lagrangean description of the (critical) string – the simplest scenario

X :  $\hookrightarrow$ 

(Σ, γ) $\qquad$ $(M, \mathrm{g}, \mathcal{G}),\ \mathrm{curv}(\mathcal{G}) =: \mathrm{H} \in Z^3(M)$

governed by the action functional $(\mathrm{d}X \underset{\mathrm{loc.}}{=} \partial_a X^\mu\, \mathrm{d}\sigma^a \otimes \partial_\mu)$

$$S_\sigma[X;\gamma] = -\tfrac{1}{2} \int_\Sigma \mathrm{g}(\mathrm{d}X \overset{\wedge}{,} \star_\gamma \mathrm{d}X) - \mathrm{i}\,\log \mathrm{Hol}_{\mathcal{G}}(X)$$

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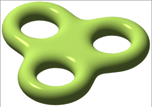
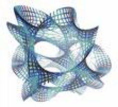
$X \quad :$  $\hookrightarrow$ 

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$\mathrm{Hol}_{\mathcal{G}}(X)$ is the SURFACE HOLONOMY of GERBE $\mathcal{G}$, locally given by

$$\begin{cases} \mathrm{H}|_{\mathcal{O}_i} =: \mathrm{d}B_i \\ (B_j - B_i)|_{\mathcal{O}_i \cap \mathcal{O}_j} =: \mathrm{d}A_{ij} \\ (A_{jk} - A_{ik} + A_{ij})|_{\mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_k} =: \mathrm{i}\, \mathrm{d} \log g_{ijk} \\ (g_{jkl} \cdot g_{ikl}^{-1} \cdot g_{ijl} \cdot g_{ijk}^{-1})|_{\mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_k \cap \mathcal{O}_l} = 1 \end{cases} \quad \mathrm{mod} \quad \begin{cases} B_i \mapsto B_i + \mathrm{d}\Pi_i \\ A_{ij} \mapsto A_{ij} + (\Pi_j - \Pi_i)|_{\mathcal{O}_i \cap \mathcal{O}_j} - \mathrm{i}\, \mathrm{d} \log \chi_{ij} \\ g_{ijk} \mapsto g_{ijk} \cdot (\chi_{jk}^{-1} \cdot \chi_{ik} \cdot \chi_{ij}^{-1})|_{\mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_k} \end{cases}$$

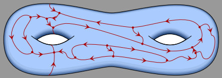
Pour qu'on n'en ait pas (que) la gerbe. . .

La gerbe, Henri Matisse (1953)

The string background for the multiphase world-sheet

Natural generalisation (e.g., strings on orbifolds and T-folds):

Σ with embedded DEFECT Γ



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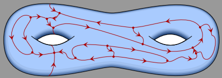
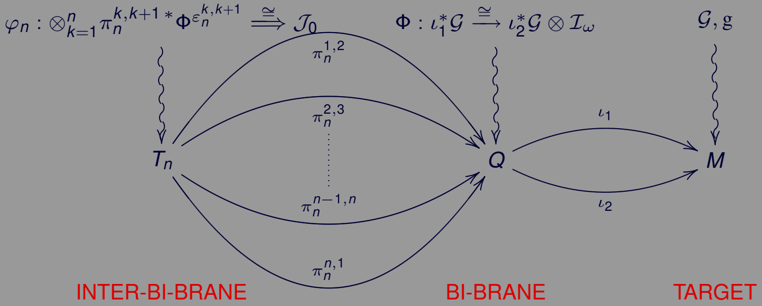


σ-model requires STRING BACKGROUND $\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$



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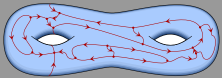

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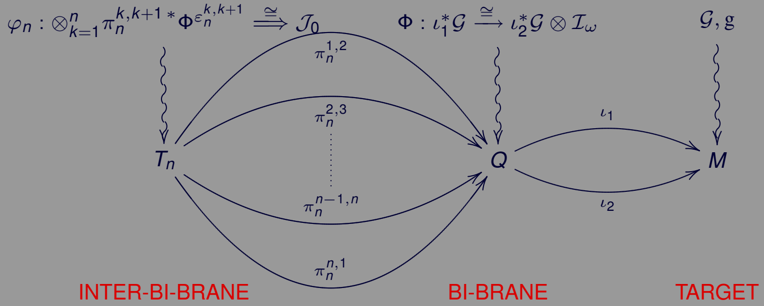

Example: D-brane $\equiv$ boundary bi-brane for $M = \mathscr{M} \sqcup \{\bullet\}$,

$$\iota_1 \;:\; Q = \mathscr{D} \hookrightarrow \mathscr{M}\,, \qquad \iota_2 \;:\; \mathscr{D} \to \{\bullet\}\,, \qquad \Phi_\partial \;:\; \mathcal{G}|_{\mathscr{D}} \xrightarrow{\cong} \mathcal{I}_{\omega = B_i + \mathrm{d}A_i}$$

The string background for the multiphase world-sheet – ctd.

Upshot: cohomological classification scheme for σ-models

FIELD THEORY	CFT_α	$\mathrm{D}_{\alpha\beta}$	$\mathrm{J}_{\alpha_1\alpha_2\ldots\alpha_n}$
GEOMETRY	TARGET $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$	BI-BRANE $\mathcal{B} = (Q, \omega, \iota_1, \iota_2, \Phi)$	INTER-BI-BRANE $\mathcal{I} = \left(T_n, \varphi_n, (\varepsilon_n^{k,k+1}, \pi_n^{k,k+1}) \mid n \in \mathbb{N}_{>0}\right)$
2-CATEGORY	object	1-morphism	2-morphism
$\mathfrak{BGrb}(M \sqcup Q \sqcup T)$	$\mathcal{G}$	$\Phi : \iota_1^*\mathcal{G} \xrightarrow{\sim} \iota_2^*\mathcal{G} \star \mathcal{I}(\omega)$	$\varphi_n : \circ_{k=1}^n \pi_n^{k,k+1*} \Phi^{\varepsilon_n^{k,k+1}} \xRightarrow{\sim} \mathrm{id}$
classification	$W^2(M, \mathrm{H})$ is $H^2(M, \mathrm{U}(1))$-torsor	$W^1(M, \mathcal{G}_1 \to \mathcal{G}_2)$ is $H^1(M, \mathrm{U}(1))$-torsor	$W^0(M, \Phi_1 \Rightarrow \Phi_2)$ is $\mathrm{U}(1)^{\lvert\pi_0(M)\rvert}$-torsor

Part II

Dualities via world-sheet defects

The canonical interpretation of defects – lines

Categorial quantisation & more geometric analyses suggest

$$\text{(some) DEFECTS} \qquad \sim \qquad \text{STRING DUALITIES}$$

This can be rendered rigorous in the 2-categorial setting. . .

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Thm.: $\mathfrak{B}$ canonically defines PREQUANTUM BUNDLE

$$\mathcal{L}_\sigma \to \mathsf{P}_\sigma\,, \qquad \mathrm{curv}(\mathcal{L}_\sigma) = \Omega_\sigma$$

Def.: DUALITY $\equiv$ Ω_σ^--lagrangean submanifold

$$\mathfrak{D}_\sigma \subset \mathsf{P}_\sigma \times \mathsf{P}_\sigma\,, \qquad \mathrm{pr}_1^*\mathscr{H}_\sigma = \mathrm{pr}_2^*\mathscr{H}_\sigma$$

together with a bundle isomorphism

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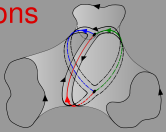
Thm.: $\mathcal{B}$ canonically defines a duality iff

- $\widetilde{\iota}_\alpha \;:\; LQ \to LM \;:\; X \mapsto \iota_\alpha \circ X$ are surjective submersions
- (Γ, X) topological
- extra conditions (technical)

The canonical interpretation of defects – junctions

Defect junctions naturally associated with interactions



Further hints from categorial quantisation & study of maximally symmetric WZW defects.

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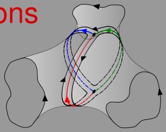

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Thm.: $(\mathcal{B}, \mathcal{J})$ can. define a Ω_σ^{+-}-isotropic INTERACTION SUBSPACE

$$\mathfrak{I}_\sigma(\circledast\mathcal{B} : \mathcal{J} : \mathcal{B}) \subset \mathsf{P}_\sigma \times \mathsf{P}_\sigma \times \mathsf{P}_\sigma$$

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$$\big(\mathrm{pr}_1^*\mathcal{L}_\sigma \otimes \mathrm{pr}_2^*\mathcal{L}_\sigma\big)|_{\mathfrak{I}_\sigma(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})} \xrightarrow{\cong} \mathrm{pr}_3^*\mathcal{L}_\sigma|_{\mathfrak{I}_\sigma(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})}$$

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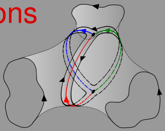

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Example: 2-iso's for maxym G-WZW defects $\sim$
spaces of conformal blocks on punctured and decorated $\mathbb{C}P^1$ via $\mathrm{CS_k(G)}$ on $\mathbb{R} \times \mathbb{C}P^1_{\{P_k\}_{k\in\overline{1,n}}}$ with parallel Wilson lines of fixed holonomy

Symmetries as distinguished dualities

Converse result for a class of dualities with local data

$$\Phi_\sigma \;:\; \mathrm{pr}_1^*\mathcal{L}_\sigma|_{\mathfrak{D}_\sigma} \xrightarrow{\cong} \mathrm{pr}_2^*\mathcal{L}_\sigma|_{\mathfrak{D}_\sigma}\;,$$

$$\mathsf{i}\,\log\Phi_{\sigma\,\mathrm{i}}[(X_1,\mathsf{p}_1),(X_2,\mathsf{p}_2)] = \int_{\mathbb{S}^1} \mathrm{Vol}(\mathbb{S}^1)\,\mathsf{p}_{2\,\mu}\,F^\mu(X_1) + W_{\mathrm{i}}[(X_1,X_2)]$$

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Thm.: Φ_σ can. defines a flat bi-brane with

- world-volume $Q = (\mathrm{id}_M \times F)(M) \subset M \times M$, with $F \in \mathrm{Isom}(M,\mathrm{g})$;
- bi-brane maps $\iota_\alpha = \mathrm{pr}_\alpha,\ \alpha \in \{1,2\}$;
- bi-brane 1-isomorphism $\Phi : \mathcal{G} \xrightarrow{\cong} F^*\mathcal{G}$

These are INTERNAL SYMMETRIES of the closed string.

Part III

Generalised geometry with a 2-categorial twist

Brackets on the state space and on the target space

Observation: GENERALISED GEOMETRY natural in the symplectic setting (P,Ω), via HAMILTONIAN SECTIONS:

$$\mathfrak{X}_h = \mathscr{X}_h \oplus h \in \ker \mathrm{d}_\Omega \subset \Gamma(\mathrm{E}^{(1,0)}\mathrm{P})\,, \qquad \mathrm{E}^{(1,0)}\mathrm{P} := \wedge^1 \mathrm{TP} \oplus \wedge^0 \mathrm{T}^*\mathrm{P} \to \mathrm{P}$$

and Ω-TWISTED VINOGRADOV BRACKET:

$$[\,\mathfrak{X}_{h_1}\,,\,\mathfrak{X}_{h_2}\,]^\Omega_{\mathrm{V}} := [\,\mathscr{X}_{h_1}\,,\,\mathscr{X}_{h_2}\,] \oplus (\mathscr{X}_{h_1} \lrcorner \mathrm{d}h_2 - \mathscr{X}_{h_2} \lrcorner \mathrm{d}h_1 + \mathscr{X}_{h_1} \lrcorner \mathscr{X}_{h_2} \lrcorner \Omega) \equiv \mathfrak{X}_{\{\,h_1\,,\,h_2\,\}_\Omega}$$

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Idea: Given the manifold structure on the space of σ-model fields, we can look for bracket structures on $\mathrm{E}^{(1,\bullet)}(M \sqcup Q \sqcup T) \to M \sqcup Q \sqcup T$ **closing** on σ-SYMMETRIC SECTIONS, with a **homomorphic lift** to

CANONICAL VINOGRADOV STRUCTURE

$$\mathfrak{V}^{\Omega}\mathrm{P}_\sigma := \left(\mathrm{E}^{(1,0)}\mathrm{P}_\sigma, [\,\cdot\,,\,\cdot\,]^{\Omega_\sigma}_{\mathrm{V}}\,, \alpha_{\mathrm{TP}_\sigma}\right)$$

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Hint: The answer is known for $\mathrm{P}_{\sigma,\emptyset}$: Courant algebroid $\mathfrak{C}^{\mathrm{H}}M$ on $\mathrm{E}^{(1,1)}M$ with Courant bracket twisted by H à la Ševera–Weinstein, Hitchin-isomorphic with $\mathfrak{C}_{\mathcal{G}}M$

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Observation: Study of automorphisms and Hitchin-type iso's

$$\mathfrak{M}^{(1,0),(\mathrm{H},\omega;\Delta_Q)}_{\iota_\alpha}(M \sqcup Q) \cong \mathfrak{M}^{(1,0),(0,0;\Delta_Q)}_{(\mathcal{G},\mathcal{B}),\iota_\alpha}(M \sqcup Q)$$

indicate that generalised geometry is a natural generalisation of the geometry of $\mathsf{T}M$ in the presence of $\mathfrak{BGrb}^\nabla(M \sqcup Q \sqcup T)$

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$$(\,\mathfrak{V}_1\,,\,\mathfrak{V}_2\,)_\lrcorner\; = \tfrac{1}{2}\,({}^M\mathscr{V}_1\lrcorner\,\upsilon_2 + {}^M\mathscr{V}_2\lrcorner\,\upsilon_1)$$

Observation: Study of automorphisms and Hitchin-type iso's

$$\mathfrak{M}_{\iota_\alpha}^{(1,0),(\mathrm{H},\omega;\Delta_Q)}(M \sqcup Q) \cong \mathfrak{M}_{(\mathcal{G},\mathcal{B}),\iota_\alpha}^{(1,0),(0,0;\Delta_Q)}(M \sqcup Q)$$

indicate that generalised geometry is a natural generalisation of the geometry of $\mathsf{T}M$ in the presence of $\mathfrak{BGrb}^\nabla(M \sqcup Q \sqcup T)$

In keeping with the above,

Thm.: $\mathcal{B}$ can. induces a morphism in the category of twisted Courant algebroids on $\mathsf{E}^{(1,1)}Q$.

Gaugeability constraints for σ-model symmetries

Internal symmetries of $S_\sigma \equiv$ σ-SYMMETRIC ι_α-ALIGNED SECTIONS

$$\mathfrak{K}_a = ({}^M\mathfrak{K}_a, {}^Q\mathfrak{K}_a) = ({}^M\mathscr{K}_a \oplus \kappa_a, {}^Q\mathscr{K}_a \oplus k_a)\,, \qquad a \in \overline{1, \dim \mathfrak{k}_{\sigma,\iota_\alpha}}$$

$$\begin{cases} \iota_{\alpha\,*}{}^Q\mathscr{K}_a = {}^M\mathscr{K}_a\big|_{\iota_\alpha(Q)} \\ \mathrm{d}_{\mathrm{H}}{}^M\mathfrak{K}_a = 0 \\ \mathrm{d}_\omega{}^Q\mathfrak{K}_a + \Delta_Q\kappa_a = 0 \end{cases}$$

The corresponding hamiltonian sections

$$\widetilde{\mathfrak{K}}_a = \mathrm{e}^{\mathrm{pr}_{\mathsf{T}^*\mathsf{L}_\sigma M}\theta_{\mathsf{T}^*\mathsf{L}_\sigma M}} \triangleright \widetilde{L}\mathfrak{K}_a\,, \qquad \widetilde{L}\;:\;\Gamma_{\iota_\alpha}\big(\mathsf{E}^{(1,1)}M \sqcup \mathsf{E}^{(1,0)}Q\big) \to \Gamma\big(\mathsf{E}^{(1,0)}\mathsf{P}_\sigma\big)\,,$$

written in terms of the canonical 1-form $\theta_{\mathsf{T}^*\mathsf{L}_\sigma M} \in \Omega^1(\mathsf{T}^*\mathsf{L}_\sigma M)$, obey

$$\Big[\,\widetilde{\mathfrak{K}}_a\,,\,\widetilde{\mathfrak{K}}_b\,\Big]^{\Omega_\sigma}_{\mathrm{V}} = \widetilde{[[\,\mathfrak{K}_a\,,\,\mathfrak{K}_b\,]]}^{(\mathrm{H},\omega;\Delta_Q)}$$

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The realisation of $\mathfrak{k}_{\sigma,\iota_\alpha}$ on the state space becomes hamiltonian iff

$$[[\,\mathfrak{K}_a\,,\,\mathfrak{K}_b\,]]^{(\mathrm{H},\omega;\Delta_Q)} = f_{ab}{}^c\,\mathfrak{K}_c$$

Gaugeability constraints for σ-model symmetries – ctd.

Conclusion: Necessary conditions of gaugeability of $\mathfrak{k}_{\sigma,\iota_\alpha}$:

$$\big(\bigoplus_{a\in\overline{1,\dim\mathfrak{k}_{\sigma,\iota_\alpha}}}\mathbb{R}\,\mathfrak{K}_a,[[\cdot\,,\,\cdot]]^{(\mathrm{H},\omega;\Delta_Q)}\big)\cong\mathfrak{k}_{\sigma,\iota_\alpha}\qquad\wedge\qquad(\,\mathfrak{K}_a\,,\,\mathfrak{K}_b\,)_\lrcorner=0$$

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N.B. The ‘isotropy’ condition nullifies the anomaly of the Poisson algebra of the Noether currents.

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Equivalently, the gaugeability relations ensure the existence of a $\mathfrak{k}_{\sigma,\iota_\alpha}$-equivariantly closed extension $\widehat{\mathrm{H}}$ of H, and a $\mathfrak{k}_{\sigma,\iota_\alpha}$-equivariant extension $\widehat{\omega}$ of ω in the Cartan model of $\mathfrak{k}_{\sigma,\iota_\alpha}$-equivariant cohomology of $M\sqcup Q$, s.t.

$$\widehat{\mathrm{d}}\widehat{\mathrm{H}}=0\,,\qquad\qquad\widehat{\mathrm{d}}\widehat{\omega}=-\Delta_Q\widehat{\mathrm{H}}$$

for $\widehat{\mathrm{d}}\eta(X)=\mathrm{d}\eta(X)+X^a\,\mathscr{K}_a\lrcorner\,\eta(X),\ X\in\mathfrak{k}_{\sigma,\iota_\alpha}$.

Part IV

The gauged σ-model

Motivation for & problems with gauging

I Motivation:

I.1 string theory on cosets (via gauging & symplectic reduction);

I.2 T-duality etc.

II Problems:

II.1 lifting the geometric action of the isometry (sub-)group to $\mathfrak{BGrb}^\nabla(M \sqcup Q \sqcup T)$;

II.2 coupling (non-trivial) world-sheet gauge fields to $\mathfrak{BGrb}^\nabla(M \sqcup Q \sqcup T)$.

Equivariant structures

Simplicial descent schemes: in the absence of defects,

$$\cdots \overset{{}^{\mathrm{G}}d_i^{(4)}}{\Rrightarrow} \mathrm{G}^3 \times M \overset{{}^{\mathrm{G}}d_i^{(3)}}{\Rrightarrow} \mathrm{G}^2 \times M \overset{{}^{\mathrm{G}}d_i^{(2)}}{\Rrightarrow} \mathrm{G} \times M \overset{{}^{\mathrm{G}}d_i^{(1)}}{\rightrightarrows} M \Big(\overset{{}^{M}\varpi_{\mathrm{G}}}{- - \to} M/\mathrm{G} \Big) ,$$

$$\delta_{\mathrm{G}}\gamma = 1 \qquad \gamma \qquad \Upsilon \qquad \mathcal{G} \qquad \overline{\mathcal{G}}$$

associativity distributiveness element-wise

where

$$\Upsilon \;:\; {}^{\mathrm{G}}d_1^{(1)\,*}\mathcal{G} \xrightarrow{\cong} d_0^{(1)\,*}\mathcal{G} \otimes \mathcal{I}_\rho \,, \qquad \gamma \;:\; \big({}^{\mathrm{G}}d_0^{(2)\,*}\Upsilon \otimes \mathrm{id}\big) \circ {}^{\mathrm{G}}d_2^{(2)\,*}\Upsilon \overset{\cong}{\Longrightarrow} {}^{\mathrm{G}}d_1^{(2)\,*}\Upsilon$$

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Observation: Gaugeability fixes $\rho = \theta_L^a \wedge \kappa_a + \frac{1}{2}\,\theta_L^a \wedge \theta_L^b \left({}^M\mathscr{K}_a \lrcorner\, \kappa_b\right)$

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$\delta_{\mathrm{G}}\gamma = 1$ (associativity) $\quad \gamma$ (distributiveness) $\quad \Upsilon$ (element-wise) $\quad \mathcal{G} \quad \overline{\mathcal{G}}$

where

$$\Upsilon \;:\; {}^{\mathrm{G}}d_1^{(1)\,*}\mathcal{G} \xrightarrow{\cong} d_0^{(1)\,*}\mathcal{G} \otimes \mathcal{I}_\rho\,, \qquad \gamma \;:\; \left({}^{\mathrm{G}}d_0^{(2)\,*}\Upsilon \otimes \mathrm{id}\right) \circ {}^{\mathrm{G}}d_2^{(2)\,*}\Upsilon \overset{\cong}{\Longrightarrow} {}^{\mathrm{G}}d_1^{(2)\,*}\Upsilon$$

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‘Problem’:

Thm.: $\quad \mathfrak{BGrb}^\nabla(M)^{\mathrm{G},\rho=0} \;\cong\; \mathfrak{BGrb}^\nabla(M/\mathrm{G})$

However, $\kappa_a \not\equiv 0$ generically.

Equivariant structures – ctd.

Similarly, in the presence of defects,

$$\cdots \overset{{}^{\mathrm{G}}d_i^{(3)}}{\ggg} \mathrm{G}^2 \times Q \overset{{}^{\mathrm{G}}d_i^{(2)}}{\rightrightarrows} \mathrm{G} \times Q \overset{{}^{\mathrm{G}}d_i^{(1)}}{\rightrightarrows} Q \left(- \overset{{}^{Q}\varpi_{\mathrm{G}}}{-\,-\!\!\to} Q/\mathrm{G} \right),$$

$$\mathrm{c.c.}(\Xi, \gamma) \qquad \Xi \qquad \Phi \qquad \overline{\Phi}$$

where

$$\Xi \ : \ \left(\iota_2^* \Upsilon \otimes \mathrm{id}\right) \circ {}^{Q}d_1^{(1)\,*}\Phi \overset{\cong}{\Longrightarrow} \left({}^{Q}d_0^{(1)\,*}\Phi \otimes \mathcal{J}_\lambda\right) \circ \iota_1^* \Upsilon$$

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Observation: Gaugeability fixes $\lambda = \theta_L^a\, k_a$

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'Problem':

Thm.: $\left\{ \begin{array}{c} \text{Equivalence classes} \\ \text{of } (\mathrm{G}, 0)\text{-equivariant bi-branes} \\ \text{with world-volume } Q \\ \text{for } (\mathrm{G}, 0)\text{-equivariant gerbes over } M \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Equivalence classes} \\ \text{of bi-branes} \\ \text{with world-volume } Q/\mathrm{G} \\ \text{for gerbes over } M/\mathrm{G} \end{array} \right\} .$

However, $k_a \not\equiv 0$ generically.

The coupling of the world-sheet gauge field

Observation: Gauging G prerequires replacing $X \in C^1(\Sigma, M \sqcup Q)$ with $X \in \Gamma\big(P \times_{\mathrm{G}} (M \sqcup Q)\big)$ for

principal G-bundle $\mathrm{G} \hookrightarrow P \to \Sigma\,,$ with principal G-connection $\mathcal{A} \in \Omega^1(P) \otimes \mathfrak{g}$

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The study of the conditions of invariance (for P trivial) of

$$S_{\sigma,\mathrm{top}}[(\Gamma, X)] \mapsto S_{\sigma,\mathrm{top}}[(\Gamma, X)] + \int_\Sigma (X \times \mathrm{id}_\Sigma)^* \zeta_A + \int_\Gamma \big((X \times \mathrm{id}_\Gamma)|_\Gamma\big)^* \mu_A\,,$$

$$\zeta_A(\sigma, m) = -\alpha_a(m) \wedge A^a(\sigma) + \tfrac{1}{2}\,\beta_{ab}(m)\,A^a \wedge A^b(\sigma)\,, \qquad \mu_A(\sigma, m) = \gamma_a(m)\,A^a(\sigma)$$

leads to the definitions

$$\mathcal{G}_{\mathcal{A}} \quad := \quad \mathrm{pr}_2^* \mathcal{G} \otimes \mathcal{I}_{\rho_{\mathcal{A}}}\,, \qquad \rho_{\mathcal{A}}(p, m) := -\kappa_a(m) \wedge \mathcal{A}^a(p) + \tfrac{1}{2}\,({}^M\mathscr{K}_a \lrcorner\, \kappa_b)(m)\,\mathcal{A}^a \wedge \mathcal{A}^b(p)\,,$$

$$\Phi_{\mathcal{A}} \quad := \quad \mathrm{pr}_2^* \Phi \otimes \mathcal{J}_{\lambda_{\mathcal{A}}}\,, \qquad \lambda_{\mathcal{A}}(p, m) := k_a(m)\,\mathcal{A}^a(p)$$

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$$\Phi_{\mathcal{A}} := \mathrm{pr}_2^* \Phi \otimes \mathcal{J}_{\lambda_{\mathcal{A}}}\,, \qquad \lambda_{\mathcal{A}}(p, m) := k_a(m)\, \mathcal{A}^a(p)$$

Thm.: $\mathcal{G}_{\mathcal{A}}$ carries a canonical structure of a $(\mathrm{G}, 0)$-equivariant gerbe on $P \times M$, and $\Phi_{\mathcal{A}}$ carries a canonical structure of a $(\mathrm{G}, 0)$-equivariant $\mathcal{G}_{\mathcal{A}}$-bi-brane on $P|_\Gamma \times Q$.

The coupling of the world-sheet gauge field – ctd.

Corollary: $(\mathcal{G}_{\mathcal{A}}, \Phi_{\mathcal{A}})$ descend to unique (equivalence classes of) $(\overline{\mathcal{G}}_{\mathcal{A}}, \overline{\Phi}_{\mathcal{A}})$ over $P \times_{\mathrm{G}} (M \sqcup Q)$, and so can be used to define the G-GAUGED σ-MODEL

$$S_\sigma[(\Gamma, X); \mathcal{A}, \gamma] = S_{\sigma,\mathrm{kin}}[X; \mathcal{A}, \gamma] - \mathrm{i} \log \mathrm{Hol}_{\overline{\mathcal{G}}_{\mathcal{A}}, \overline{\Phi}_{\mathcal{A}}}(\Gamma, X)\,,$$

with $S_{\sigma,\mathrm{kin}}[X; \mathcal{A}, \gamma]$ obtained through minimal coupling.

The coupling of the world-sheet gauge field – ctd.

Corollary: $(\mathcal{G}_{\mathcal{A}}, \Phi_{\mathcal{A}})$ descend to unique (equivalence classes of) $(\overline{\mathcal{G}}_{\mathcal{A}}, \overline{\Phi}_{\mathcal{A}})$ over $P \times_{\mathrm{G}} (M \sqcup Q)$, and so can be used to define the G-GAUGED σ-MODEL

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with $S_{\sigma,\mathrm{kin}}[X; \mathcal{A}, \gamma]$ obtained through minimal coupling.

Recall that the gauge group G_Σ is the set $\Gamma(P \times_{\mathrm{Ad\,G}} \mathrm{G})$ with the group operation induced from

$$[(p, g_1)] \cdot [(p, g_2)] := [(p, g_1 \cdot g_2)]$$

and with the action on P induced by

$$(P \times_{\mathrm{Ad\,G}} \mathrm{G}) \times P \to P \ : \ \big([(\tau_i(\sigma, g), h)], \tau_i(\sigma, g)\big) \mapsto \tau_i(\sigma, h \cdot g) =: [(\tau_i(\sigma, g), h)] \rhd \tau_i(\sigma, g) \,.$$

as per

$$\lambda_\cdot \ : \ \Gamma(P \times_{\mathrm{Ad\,G}} \mathrm{G}) \times P \to P \ : \ \big(\chi_i, \tau_i(\sigma, g)\big) \mapsto \chi_i(\sigma) \rhd \tau_i(\sigma, g) =: \lambda_{(\chi_i)}\big(\tau_i(\sigma, g)\big) \,.$$

The coupling of the world-sheet gauge field – ctd.

We have the fundamental

Thm.: $S_\sigma[(\Gamma, X); \mathcal{A}, \gamma]$ is invariant under G-gauge transformations

$$\big((\chi_i), X\big) \mapsto \big((\lambda_{(\chi_i)} \circ \mathrm{pr}_1) \times \mathrm{pr}_2\big) \circ X\,, \qquad \mathcal{A} \mapsto \lambda^*_{(\chi_i^{-1})}\mathcal{A}\,.$$

Proof: uses the G-equivariance of $\mathcal{A}$ and the form of $\rho_{\mathcal{A}}$ and $\lambda_{\mathcal{A}}$.

Part V

Outlook

Outlook

- understanding T-duality, with particular emphasis on geometric structures behind the metric, the torsion and the dilaton;
- construction of spaces modelled on toroidal bundles only locally;
- including supersymmetry in the generalised geometric framework with a 2-categorial twist;
- study of the effective gerbe-twisted gauge field theory and the emergent geometry of bi-branes in the gerbe-theoretic context;
- gerbe theory vs criticality (generalised Ricci flows?);
- ‘holographic principle’ for higher categorial structures;
- …