Partition Functions via Quasinormal Mode Methods: Spin, Product Spaces, and Boundary Conditions

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Review: Field Theory Objects

History

- Partition function $Z[\phi]$
- Effective Action $-\log Z$ or S_{eff}
- One-loop determinant

$$\frac{1}{\det \nabla^2} = Z[\phi]$$

- Effective potential (Legendre transform)

We will mainly focus on Effective Actions although what we really calculate is the one-loop determinant.

A Use of the Effective Action

Quantum Entropy Function

- Classical black hole entropy:

$$\frac{A}{4G_N} = S_{BH} = S_{micro} = \log d_{micro}$$

- Higher curvature gravity: Wald entropy
- Quantum fluctuations of fields in the black hole background extremal black holes: near horizon AdS_2 with cutoff scale r_0

$$Z_{AdS_2} = Z_{CFT_1} = \text{Tr}\left[\exp\left(-2\pi r_0 H + \mathcal{O}(r_0^- 2)\right)\right]$$
$$Z_{AdS_2} \approx d_0 \exp\left(-2\pi E_0 r_0\right)$$

where d_0 is the degeneracy of the ground state.

The effective action of quantum fields in an AdS_2 background tells us the quantum contribution to the entropy of extremal black holes.

Finding the Effective Action

Possible Calculation Methods

1. Curvature Heat Kernel Expansion
2. Eigenfunction Heat Kernel method
3. Group Theory
4. Quasinormal Mode method

$$\log\det(D+m^2) = \mathrm{Tr}\log(D+m^2) = -\int_\epsilon^\infty \frac{dt}{t}\mathrm{Tr}\, e^{-t(D\ +m^2)}$$

$$= -(4\pi)^{-n/2}\sum_{k=0}^{n} a_k(D)\int_\epsilon^\infty \frac{dt}{t}t^{(k-n)/2}e^{-m^2t} + \mathcal{O}(m^-1)$$

Here n is the number of dimensions, and the a_0 are known in terms of curvature invariants, e.g. Ricci curvature R. But this only gives the determinant up to $\mathcal{O}(m^{-1})$. If we care about massless behavior it doesn't help!

Finding the Effective Action

Possible Calculation Methods

1. Curvature Heat Kernel Expansion
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$$\log\det(D) = -\int_\epsilon^\infty \frac{dt}{t}\sum_n e^{-\kappa_n t} = -\int_\epsilon^\infty \frac{dt}{t}\int d^4x\sqrt{g}K^s(x,x;t)$$

$$K^s(x,x';t) = \sum_n e^{-\kappa_n t} f_n(x) f_n^*(x')$$

where κ_n are the eigenvalues of a complete set of states with eigenfunctions f_n. (Sen, Mandal, Banerjee, Gupta, . . . 2010)
Ok for scalar, but hard for general graviton, gravitino, or even vector coupled to flux background.

Finding the Effective Action

Possible Calculation Methods

1. Curvature Heat Kernel Expansion
2. Eigenfunction Heat Kernel method
3. Group Theory
4. Quasinormal Mode method

Can we count the effect of all of these fields in another way? Yes, for sufficient supersymmetry, e.g. $\mathcal{N} = 2$! (CK, Larsen, Lisbão 2014)
What about cases with lower Susy, e.g. De Sitter with a scalar?
Also Gopakumar et. al.

Finding the Effective Action

Possible Calculation Methods

1. Curvature Heat Kernel Expansion
2. Eigenfunction Heat Kernel method
3. Group Theory
4. Quasinormal Mode method

Finding $Z(m^2)$

- Consider Z as a meromorphic function of m^2
- let m^2 wander the complex plane
- find poles + zeros + "behavior at infinity"

This is sufficient to know the function Z (at one loop).
(Denef, Hartnoll, Sachdev, 0908.2657; see also Coleman)

Weierstrass factorization theorem

Theorem

Any meromorphic function can be written as as a product over its poles and zeros, multiplied by an entire function:

$$f(z) = \exp Poly(z) \prod_{zeros} (z - z_0)^{d_0} \prod_{poles} \frac{1}{(z - z_p)^{d_p}}$$

Examples

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right)$$

De Sitter

Two-dimensional de Sitter space Wick-rotates to the sphere. We set the scale to a. Poles are at masses where we can solve the equations of motion, as well as periodicity.

Equations of motion and Periodicity

$$\left[\nabla^2 + m^2\right]\phi = 0$$

ϕ is just our usual spherical harmonic Y_{lm}, so when $-m^2 = \frac{l^*(l^*+1)}{a^2}$ and l^* is an integer.

So poles are when

$$l^* = \frac{1}{2} \pm i\sqrt{m^2a^2 - \frac{1}{4}}$$

is an integer, and the degeneracy of each pole is $2l^* + 1$.

De Sitter

Using these poles and degeneracies we have

$$\log Z_{dS_2} = \log \det \nabla^2_{dS_2} = Poly + \sum_{\pm, n \geq 0} (2n+1) \log(n + l^* \pm).$$

where we have

$$l^* = \frac{1}{2} \pm i\sqrt{m^2 a^2 - \frac{1}{4}} \equiv \frac{1}{2} \pm i\nu.$$

We can regularize using (Hurwitz) zeta functions:

$$\log Z^{complexscalar}_{dS^2} - Poly = \sum_{\pm} \left[2\zeta'\left(-1, l^*_{\pm}\right) - \left(2l^*_{\pm} - 1\right)\zeta'\left(0, l^*_{\pm}\right)\right]$$

$$\approx \left(\log \nu^2 - 3\right)\nu^2 - \frac{1}{12}\log \nu^2 + \mathcal{O}(\nu^{-1})$$

where

$$\zeta(s, x) = \sum_{n=0}^{\infty} (n + x)^{-s}, \quad \zeta' = \partial_s \zeta.$$

De Sitter

Now expand using curvature heat kernel (it can get up to m^{-1}):

$$\mathcal{O}(\frac{1}{\nu}) + \log Z_{dS^2}^{complexscalar} - Poly \approx \left(\log \nu^2 - 3\right)\nu^2 - \frac{1}{12}\log \nu^2$$

$$\left(\nu^2 - \frac{1}{12}\right)\log \frac{\nu^2}{a^2\Lambda^2} - \nu^2 + \mathcal{O}(\frac{1}{\nu}) - Poly = \left(\log \nu^2 - 3\right)\nu^2 - \frac{1}{12}\log \nu^2$$

$$-Poly = -2\nu^2 + \left(\nu^2 - \frac{1}{12}\right)\log a^2\Lambda^2.$$

Note $Poly$ really is polynomial in ν!

Result: One-loop Partition Function for Complex Scalar on de Sitter in Two Dimensions

$$\log Z_{dS^2} = 2\nu^2 + \sum_{\pm}\left[2\zeta'\left(-1, l_{\pm}^*\right) - \left(2l_{\pm}^* - 1\right)\zeta'\left(0, l_{\pm}^*\right)\right] + \Lambda \text{ terms}$$

Note the cutoff regulation terms of the form $\log \Lambda$ here; they arose from the heat kernel curvature expansion.

Quasinormal Mode Method

Ingredients we need

- direction w/ periodicity or a quantization constraint
- analyticity (meromorphicity) of Z
- locations/multiplicities of zeros/poles in complex mass plane
- extra info to find $Poly$ (behavior at large mass)

Why Quasinormal modes?

In a general (thermal) spacetime, 'good' ϕ are regular and smooth everywhere in Euclidean space, where $\tau_E \sim \tau_E + 1/T$.

Euclidean 'good' ϕ

- normalizable at boundary of spacetime
- regular at origin: Pick coordinates $u = \rho e^{i\theta}$.

$$\text{for } n \geq 0,\ \phi \sim u^n = \rho^n e^{-in\theta} = \rho^{\omega_n/2\pi T} e^{-i\omega_n \tau}$$

$$\text{for } n \leq 0,\ \phi \sim \bar{u}^n = \rho^{-n} e^{-in\theta} = \rho^{-\omega_n/2\pi T} e^{i\omega_n \tau}$$

Wick rotate ϕ for $n \geq 0$, and we obtain quasinormal mode with frequency ω_n:

$$\phi \sim \left(\rho^{1/2\pi T}\right)^{-i(i\omega_n)} e^{-i(i\omega_n)t} \sim e^{-i(i\omega_n)(x+t)}.$$

Ingoing mode, using $x = \log \rho / 2\pi T$.

Why Quasinormal modes?

Quasinormal modes

- normalizable at boundary, ingoing at horizon.
- physical modes at real mass values, but imaginary frequencies
- e.g. for de Sitter,

$$-i\frac{2k + l + \frac{1}{2} \pm \nu}{a} = 2\pi i n T$$

- useful for black hole evolution, so known for many black holes and other spacetimes

Method review

Applying the Quasinormal Mode Method

1. assume partition function is meromorphic function of mass parameter $Z(\tilde{m})$
2. continue mass parameter $\tilde{m}$ to complex plane
3. find poles: mass parameter values where there is a ϕ that solves both EOMs and periodicity+boundary conditions
4. zeta function regularize sum over poles
5. use curvature heat kernel to get large mass behavior
6. compare to zeta sum large mass behavior to find $Poly$

If Poly is actually a polynomial, then that is a nontrivial check that all poles have been included (and the function is actually meromorphic).

Anti De Sitter

Scalars in even-dimensional AdS

- In AdS, we must set boundary conditions to be $r^{-\Delta}$ rather than "normalizeable".
- The special ϕ we are interested in occur at negative integer values of Δ, so they blow up at the boundary as some integer power of r. They are not normalizable in our usual sense, but still produce the correct poles in the complex-mass partition function.

These special ϕ can also be interpreted as finite representations of $SL(2,R)$.

Anti De Sitter via representations

$SL(2, R)$ scalar representations

- $SL(2, R)$ is isometry group of AdS$_2$, with generators L_0, $L_\pm$
- Label states by their eigenvalues under the Casimir (Δ) and L_0
- $L_\pm$ act as raising/lowering operators for L_0 eigenvalue

Representations have fixed Δ; we want only finite length reps (multiplicity of pole should be finite). Thus they should have both a highest and lowest weight state, so the highest weight state $|h\rangle$ has:

1. $L_+|h\rangle = 0$
2. $L_-^k|h\rangle = 0$, implies $k = 2h + 1$
3. $L_0|h\rangle = h|h\rangle$, casimir eigenvalue $\Delta = h$

For scalars specifically we find $h \in \mathbb{Z}_{\leq 0}$.
These states are linear combinations of the special ϕ earlier!

This method is easier to extend to spinors, vectors, and (massive) spin 2 d.o.f's.

New spaces (1): QNM argument for spin

In a general (thermal) spacetime, ‘good’ ϕ_μ are regular and smooth everywhere in Euclidean space, where $\tau_E \sim \tau_E + 1/T$.

Euclidean ‘good’ ϕ_μ

- normalizable at boundary of spacetime
- regular at origin: Correct condition is now square integrable:

$$\int \sqrt{g} g^{\mu\nu} \phi_\mu^* \phi_\nu < \infty$$

- Wick rotate ϕ_μ for for $n \geq s$, and we obtain QNM with frequency ω_n:

  for $n \geq s$, $\phi_i \sim u^n = \rho^n e^{-in\theta} = \rho^{\omega_n/2\pi T} e^{-i\omega_n \tau}$

  Here i only runs over non-radial indices. For transverse tensors, ϕ_ρ components have extra powers of $1/\rho$.

For $n < s$, some QNMs may not rotate to good Euclidean modes. Only good Euclidean modes should get counted.

New spaces (2): Warped AdS upcoming w/ A. Castro, P. Szepietowski

In a general (thermal) spacetime, 'good' ϕ are regular and smooth everywhere in Euclidean space, where $\tau_E \sim \tau_E + 1/T$.

Euclidean 'good' ϕ

- ~~normalizable~~ satisfies boundary conditions
  Compere, Song, Strominger at boundary of spacetime
- regular at origin: Pick coordinates $u = \rho e^{i\theta}$.

$$\text{for } n \geq 0,\ \phi \sim u^n = \rho^n e^{-in\theta} = \rho^{\omega_n/2\pi T} e^{-i\omega_n \tau}$$

$$\text{for } n \leq 0,\ \phi \sim \bar{u}^n = \rho^{-n} e^{-in\theta} = \rho^{-\omega_n/2\pi T} e^{i\omega_n \tau}$$

Wick rotate ϕ for $n \geq 0$, and we obtain quasinormal mode with frequency ω_n:

$$\phi \sim \left(\rho^{1/2\pi T}\right)^{-i(i\omega_n)} e^{-i(i\omega_n)t} \sim e^{-i(i\omega_n)(x+t)}.$$

Ingoing mode, using $x = \log \rho/2\pi T$.

New Spaces (3): Product Spaces upcoming w/ D. McGady

- For S^1, poles are at $m = n \in \mathbb{Z}$, with degeneracy 1.

$$\log Z = \text{Poly} + \sum_{n \in \mathbb{Z}} \log(n - m) \text{ from Hurwitz } \zeta(s, x) = \sum_n \frac{1}{(n + x)^s}.$$

- For $S^1 \times S^1$, poles are at $-m^2 = n_1^2 + n_2^2$, $(n_1, n_2) \in \mathbb{Z}$, again with degeneracy 1. Now we need Epstein-Hurwitz:

$$\zeta_{\text{EH}}(s, x) = \sum_{n_1, n_2} \frac{1}{(n_1^2 + n_2^2 + x)^s}.$$

- For $S^p \times S^q$ poles are at $-m^2 = n_1(n_1 + p - 1) + n_2(n_2 + q - 1)$, $(n_1, n_2) \in \mathbb{Z}_{\geq 0}$ with spherical harmonic degeneracies. Now we need generalized Epstein-Hurwitz and derivatives thereof:

$$\sum_{n_1 \geq 0, n_2 \geq 0} \frac{1}{(\alpha_1(n_1 + \beta_1)^2 + (\alpha_2(n_2 + \beta_2))^2 + x)^s}.$$

Future Possibilities

The Future:

- Simplicity of heat kernels in product space ($K_{1\times 2} = K_1 K_2$) vs. QNM method
- product spaces with AdS factors
- numerical QNMs: see esp. Arnold, Szepietowski, Vaman (1603.08994)
- large D spacetimes
- actions beyond just kinetic term?
- meromorphicity of Z?
- physical interpretation of zero modes