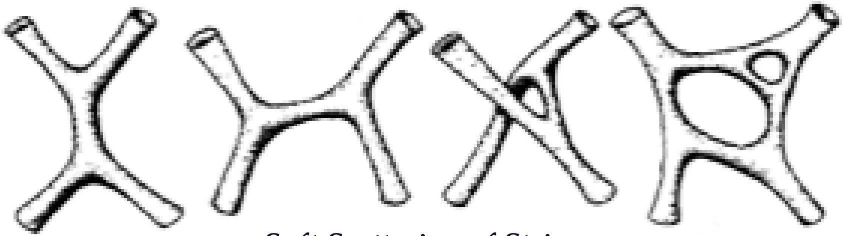

Soft Scattering of Strings

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Nordic String Meeting

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Manifestation of symmetries in physical observables

Two essentially different ways that a symmetry is realized (linear vs nonlinear)

- Manifest:
  Ground state invariant.
  Consequence:
  1) Spectrum degenerate on irred. reps.
  2) S-matrix invariant.

- Hidden: (or spontaneously broken [Nambu, Goldstone])
  Ground state not invariant. But currents still conserved!
  Consequence:
  1) Existence of (spin 0) Nambu-Goldstone (NG) bosons.
  2) **Soft theorems relating S-matrices with and without NG insertions.**

Yet another type;

- Gauge symmetry - rather a principle than symmetry:
  Necessary to reconcile SR and QM for massless spin 1 and spin 2 bosons.
  Consequence:
  1) Massless spin 1/spin 2 bosons with only two d.o.f. (in $d = 4$).
  2) **Soft theorems relating S-matrices with and without a gauge boson.**

Recent fuzz/developments

- Gauge theory soft theorems from asymptotic symmetries in GR
  [‘13, ‘14, ‘15 Strominger et al.]
- New graviton ssL soft theorem (tree-level) - ’14
  Spinor-helicity & BCFW: [Cachazo, Strominger]
  CHY formalism: [Afkhami-Jeddi], [Schwab, Volovich]
  Gauge invariance: [Broedel, De Leeuw, Plefka, Rosso], [Bern, Davies, Di Vecchia, Nohle]
- New (double) soft theorems in gauge, string, supersymmetric and effective field theories (hidden symmetries?), new collinear results, etc.

* * *

- New uses of soft theorems:
  Effective field theories from soft theorems [Weinberg ‘67]
  [Cheung, Kampf, Novotny, Trnka ‘14], [Low ‘14], [Huang, Wen ‘15], [Bianchi, Guerrieri, Huang, Lee, Wen ‘16]

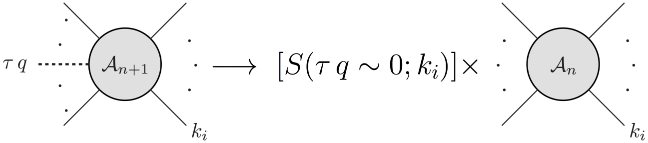
  Soft BCFW construction [Cheung, Kampf, Novotny, Shen, Trnka ‘15], [Luo, Wen ‘15]

  Extended theories from soft limits in CHY [Cachazo, Cha, Mizera ‘16]

$$A_n^{\text{theory}_1} \xrightarrow{\text{soft limit}} \tau^p A_{n-1}^{\text{theory}_1 \oplus \text{theory}_2} + \mathcal{O}(\tau^{p+1}), \quad p > 0.$$

Soft Factorization Theorems

Soft emission theorems: universal low-energy properties of amplitudes



Factorization only symmetry-dependent (universal)

$$S(\tau\, q; k_i) = \tau^{-1} S_{\mathrm{L}} + \tau^{0} S_{\mathrm{sL}} + \tau^{1} S_{\mathrm{ssL}} + \cdots + \mathcal{O}(\tau^{p})$$

Famous Examples:

- '58 Low's soft photon theorem - gauge invariance $S_{\mathrm{L}},\ S_{\mathrm{sL}}^{\mathrm{tree}}$
- '64 Weinberg's soft graviton theorem - gauge invariance $S_{\mathrm{L}},\ S_{\mathrm{sL}}^{\mathrm{tree}},\ S_{\mathrm{ssL}}^{\mathrm{tree}}$
- '65 Adler's pion zero condition - shift symmetry $S_{\mathrm{L}},\ S_{\mathrm{sL}}$
- '66 Double-soft pion theorem - coset symmetry $S_{\mathrm{L}},\ S_{\mathrm{sL}}$

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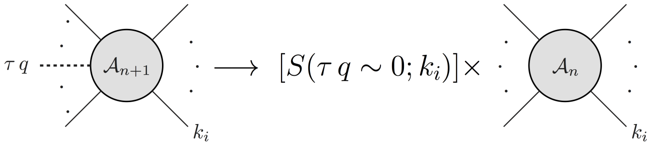

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This Talk: Scattering of soft (massless, closed) strings

- Soft emission of gravitons, dilatons and Kalb-Ramond in string theories

  Work with Paolo Di Vecchia and Raffaele Marotta:

  I. **JHEP 1505** [arXiv:1502.05258]
  II. **JHEP 1606** [arXiv:1604.03355]
  III. **JHEP 1612** [arXiv:1610.03481]

  String corrections to new graviton soft theorem,

  New dilaton soft theorem,

  New Kalb-Ramond soft theorem [unpublished]

- Soft emission of a(nother) dilaton in spont. broken conformal theories

  Work with Paolo Di Vecchia, Raffaele Marotta and Josh Nohle:

  IV. **Phys.Rev. D93** [arXiv:1512.03316]

  New (exact) theorem in spontaneously broken CFTs

Nambu-Goldstone boson of spontaneously broken conformal symmetry

- CFT with a Lorentz-scalar primary operator ξ with

$$\langle 0|\xi(0)|0\rangle \neq 0\,, \quad \Rightarrow \quad \langle 0|T^\mu_\mu|0\rangle \neq 0$$

$\mathcal{D}$ and $\mathcal{K}_\mu$ broken, $\mathcal{P}_\mu$ and $\mathcal{M}_{\mu\nu}$ (Poincaré) unbroken

- Only one Goldstone mode (dilaton) [Low, Manohar '01]

$$\langle 0|T_{\mu\nu}|\xi;q\rangle \sim q_\mu q_\nu \langle 0|\xi(0)|0\rangle\,,$$

$$\partial_\mu J^\mu_{\mathcal{D}} = T^\mu_\mu = v(-\partial^2\xi)\,, \qquad \partial_\mu J^\mu_{\mathcal{K},\rho} = 2x_\rho v(-\partial^2\xi)$$

- Ward identity (WI): (Concerning $\langle J^\mu \phi\cdots\rangle_{n+1} \equiv T^*\langle 0|J^\mu(x)\phi(x_1)\cdots\phi(x_n)|0\rangle$)

$$-iq_\mu\langle \tilde{j}^\mu(q)\tilde{\phi}\cdots\rangle_{n+1} = \langle \partial_\mu \tilde{j}^\mu\tilde{\phi}\cdots\rangle_{n+1} + \sum_{i=1}^n \langle\cdots\delta\tilde{\phi}(k_i+q)\cdots\rangle_n$$

- Soft WI:

$$\hat{f}(q)(-\partial^2)\langle\tilde{\xi}(q)\tilde{\phi}\cdots\rangle_{n+1} = -\sum_{i=1}^n\langle\cdots\delta\tilde{\phi}(k_i+q)\cdots\rangle_n + \mathcal{O}(q)$$

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Soft Theorem from Ward Identities

LSZ on WI of broken dilatation current [Callan '70, Boels&Wormsbecher '15], [IV]

$$v q^2 \langle \tilde{\xi}(q) \tilde{\phi}(k_1) \cdots \rangle_{n+1} = -i \sum_{i=1}^{n} \left(d - D - (k_i + q) \cdot \partial_{k_i} \right) \langle \cdots \tilde{\phi}(k_i + q) \cdots \rangle_n$$

$$\overset{\text{LSZ}}{\Longrightarrow} \quad v \mathcal{T}_{n+1}(q, k_i) = \left[D - nd - \sum_i k_i \cdot \partial_{k_i} - \sum_i \frac{m_i^2}{k_i \cdot q} \left(1 + q \cdot \partial_{k_i} \right) \right] \mathcal{T}_n(k_i) + \mathcal{O}(q)$$

LSZ on WI of broken $\mathcal{K}_\mu$ currents [IV]

$$-2v\partial_q^\mu \mathcal{T}_{n+1}(q, k_i) = \sum_{i=1}^{n} \left[\mathcal{K}_i^\mu - 2d\partial_{k_i}^\mu + \frac{m_i^2}{k_i \cdot q} \left(\frac{k_i^\mu}{k_i \cdot q} - \partial_{k_i}^\mu \right) \left(1 + q \cdot \partial_{k_i} + \frac{q_\nu q_\rho}{2} \partial_{k_i}^\nu \partial_{k_i}^\rho \right) \right] \mathcal{T}_n(k_i) + \mathcal{O}(q)$$

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Soft behavior completely fixed through ssL order

$$v \mathcal{T}_{n+1}(q; k_i) = \sum_{i=1}^{n} \left[\frac{D - nd}{n} - \mathcal{D}_i + \frac{q_\mu}{2} \left(\mathcal{K}_i^\mu - 2d\, \partial_{k_i}^\mu \right) \right.$$

$$\left. - \frac{m_i^2}{k_i \cdot q} \left(1 + q_\mu \partial_{k_i}^\mu + \frac{q_\nu q_\rho}{2} \partial_{k_i}^\nu \partial_{k_i}^\rho \right) \right] \mathcal{T}_n(k_i) + \mathcal{O}(q^2)$$

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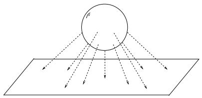
String Amplitudes and their soft limits



Tree-level string amplitude

$$\mathcal{M}_n(k_1,\ldots,k_n) \sim \int \frac{\prod_{i=1}^n dz_i}{d\Omega_{\rm M}} \langle V_1(z_1,k_1)\cdots V_n(z_n,k_n)\rangle [\otimes c.c.]_{\rm closed}$$

Suitable for studying soft limits (of bosonic sector) [I]

$$\mathcal{M}_{n+1}(q,k_1,\ldots,k_n) \sim \underbrace{\int \frac{\prod_{i=1}^n dz_i}{d\Omega_{\rm M}} \langle V_1(z_1,k_1)\cdots V_n(z_n,k_n)\rangle}_{\mathcal{M}_n(k_1,\ldots,k_n)} \underbrace{\int dz \prod_{j=1}^n \langle V_q(z,q) V_j(z_j,k_j)\rangle}_{S_q(q,\{k_i,z_i\})}$$

Follows, since V_i can be exponentiated (bosonic sector).

Existence of a soft theorem for $q \ll k_i$ implies

$$\mathcal{M}_n * S_q(q,\{k_i,z_i\}) = \hat{S}(q,k_i)\mathcal{M}_n + \mathcal{O}(q^p)$$

Extendable to multi-soft expansions

For double-soft gluons and scalars, see [1507.00938, P. Di Vecchia, R. Marotta, M.M.]

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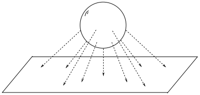

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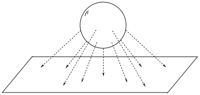

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Polarization stripped amplitude of a soft massless NS-NS closed state

- Amplitude linear in polarization vectors:

$$\mathcal{M}_{n+1}(q, k_i) = \epsilon_{q,\mu}\bar{\epsilon}_{q,\nu}\, \mathcal{M}^{\mu\nu}_{n+1}(q, k_i)$$

- Physical polarizations (KLT: open×open = closed)

$$\epsilon_q^{\mu}\bar{\epsilon}_q^{\nu} = \underbrace{\left(\epsilon_q^{(\mu}\bar{\epsilon}_q^{\nu)} - \eta_\phi^{\mu\nu}\epsilon_q\cdot\bar{\epsilon}_q\right)}_{\epsilon_g^{\mu\nu}} + \underbrace{\left(\eta_\phi^{\mu\nu}\epsilon_q\cdot\bar{\epsilon}_q\right)}_{\epsilon_\phi^{\mu\nu}} + \underbrace{\left(\epsilon_q^{[\mu}\bar{\epsilon}_q^{\nu]}\right)}_{\epsilon_B^{\mu\nu}}$$

with

$$\eta_\phi^{\mu\nu} = \frac{\eta^{\mu\nu} - q^\mu\bar{q}^\nu - q^\nu\bar{q}^\mu}{D-2}\,, \quad q\cdot\bar{q} = 1\,, \quad q^2 = \bar{q}^2 = 0\,, \quad \epsilon_q\cdot\bar{\epsilon}_q = \sqrt{D-2}$$

- Gauge invariance implies

$$q_\mu \mathcal{M}^{\mu\nu}_{n+1}(q, k_i) = q_\nu \mathcal{M}^{\mu\nu}_{n+1}(q, k_i) = 0$$

Soft Massless Closed Bosonic String graviton, dilaton, Kalb-Ramond

Simplest case: Bosonic string scattering on n closed tachyons

$$\mathcal{M}^{\mu\nu}_{n+1}(q, k_i) \sim \underbrace{\int \frac{\prod_i d^2 z_i}{d\Omega_{\mathrm{M}}} \prod_{i<j}^{n} |z_i - z_j|^{\alpha' k_i \cdot k_j}}_{\mathcal{M}_n(k_1,\ldots,k_n)} \underbrace{\int d^2 z \prod_{l=1}^{n} |z - z_l|^{\alpha' q \cdot k_l} \sum_{i,j=1}^{n} \frac{k_i^\mu k_j^\nu}{(z - z_i)(\bar{z} - \bar{z}_j)}}_{S_q^{\mu\nu}(q,\{k_i,z_i\}) \equiv \sum_{i,j} k_i^\mu k_j^\nu \, \mathcal{I}_i^j}$$

Soft-expansion of $\mathcal{I}_i^j$ (master-integral) Leading term independent of z_i - Weinberg Soft Theorem

$$\mathcal{I}_i^j \sim \frac{2}{\alpha' q \cdot k_i} \left(1 + \alpha' \sum_{j \neq i} (k_j q) \log |z_i - z_j| + \frac{(\alpha')^2}{2} \sum_{j \neq i} \sum_{k \neq i} (k_j q)(k_k q) \log |z_i - z_j| \log |z_i - z_k| \right)$$
$$+ \alpha' \sum_{j \neq i} (k_j q) \log^2 |z_i - z_j| + \log \Lambda^2 + \mathcal{O}(q^2)$$

$$\mathcal{I}_i^j \sim \sum_{m \neq i,j} \frac{\alpha' q \cdot k_m}{2} \left(\mathrm{Li}_2 \left(\frac{\bar{z}_i - \bar{z}_m}{\bar{z}_i - \bar{z}_j} \right) - \mathrm{Li}_2 \left(\frac{z_i - z_m}{z_i - z_j} \right) - 2 \log \frac{\bar{z}_m - \bar{z}_j}{\bar{z}_i - \bar{z}_j} \log \frac{|z_i - z_j|}{|z_i - z_m|} \right)$$
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Decomposed soft function

$$S_q^{\mu\nu}(q, \{k_i, z_i\}) = \sum_{i,j} k_i^{(\mu} k_j^{\nu)} \mathcal{I}_i^j + \sum_{i,j} k_i^{[\mu} k_j^{\nu]} \mathcal{I}_i^j$$

$\log \Lambda^2$-terms vanish due to momentum conservation.
Dilogs become the Bloch-Wigner Dilog appearing only in the antisymmetric part (here vanishes).

Soft Massless Closed Bosonic String graviton, dilaton, Kalb-Ramond

Simplest case: Bosonic string scattering on n closed tachyons

$$\mathcal{M}^{\mu\nu}_{n+1}(q,k_i) \sim \underbrace{\int \frac{\prod_i d^2 z_i}{d\Omega_{\rm M}} \prod_{i<j}^{n} |z_i - z_j|^{\alpha' k_i \cdot k_j}}_{\mathcal{M}_n(k_1,\ldots,k_n)} \underbrace{\int d^2 z \prod_{l=1}^{n} |z - z_l|^{\alpha' q\cdot k_l} \sum_{i,j=1}^{n} \frac{k_i^\mu k_j^\nu}{(z-z_i)(\bar z - \bar z_j)}}_{S_q^{\mu\nu}(q,\{k_i,z_i\}) \equiv \sum_{i,j} k_i^\mu k_j^\nu \mathcal{I}_i^j}$$

Soft-expansion of $\mathcal{I}_i^j$ (master-integral) Leading term independent of z_i - Weinberg Soft Theorem

$$\mathcal{I}_i^i \sim \frac{2}{\alpha' q\cdot k_i}\left(\mathbf{1} + \alpha' \sum_{j\neq i} (k_j q) \log|z_i - z_j| + \frac{(\alpha')^2}{2} \sum_{j\neq i}\sum_{k\neq i} (k_j q)(k_k q) \log|z_i - z_j| \log|z_i - z_k| \right)$$
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Soft graviton or dilaton emission from n-tachyon interaction

$$\mathcal{M}_{n+1}^{(\mu\nu)} = \mathcal{M}_n * S_q^{(\mu\nu)}(q, \{k_i, z_i\}) \stackrel{?}{=} \hat{S}^{(\mu\nu)}(q, k_i)\mathcal{M}_n + \mathcal{O}(q^2)$$

? = Yes: [I, IV]

$$\hat{S}^{(\mu\nu)}(q, k_i) = \sum_{i=1}^{n} \left[\frac{k_i^\mu k_i^\nu}{q \cdot k_i} - i \frac{k_i^\mu q_\rho L_i^{\nu\rho}}{q \cdot k_i} - \frac{1}{2} \frac{q_\rho L_i^{\mu\rho} q_\sigma L_i^{\nu\sigma}}{q \cdot k_i} + [\hat{\eta}]_i^{\mu\nu} \right]$$

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No soft α'-operator!
In the field theory limit $\alpha' \to 0$, $\mathcal{M}_n$ becomes the amplitude of n massive ϕ^3 scalars [Scherk '71].

The graviton case

$$\epsilon^g_{\mu\nu} [\hat{\eta}]_i^{\mu\nu} = 0\,.$$

An antisymmetric state with n hard tachyons

$$\mathcal{M}_{n+1}^{[\mu\nu]} = 0\,.$$

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Soft factorization of $n+1$-point amplitude of closed bosonic strings

Same procedure and problem: $\mathcal{M}_{n+1}^{\mu\nu} = \mathcal{M}_n * S^{\mu\nu} \overset{?}{=} \hat{S}^{(\mu\nu)}(q,k_i)\mathcal{M}_n + \mathcal{O}(q^2)$

$$S^{\mu\nu}(q,\{k_i,z_i\}) = \int d^2z \sum_{i=1}^{n}\left(\frac{\theta_i\epsilon_i^{\mu}}{(z-z_i)^2} + \sqrt{\frac{\alpha'}{2}}\frac{k_i^{\mu}}{z-z_i}\right)\sum_{j=1}^{n}\left(\frac{\bar\theta_j\bar\epsilon_j^{\nu}}{(\bar z-\bar z_j)^2} + \sqrt{\frac{\alpha'}{2}}\frac{k_i^{\nu}}{\bar z-\bar z_i}\right)$$
$$\times \exp\left[-\sum_{i=1}^{n}\sqrt{\frac{\alpha'}{2}}\frac{\theta_i\epsilon_i\cdot q}{z-z_i}\right]\exp\left[-\sum_{i=1}^{n}\sqrt{\frac{\alpha'}{2}}\frac{\bar\theta_i\bar\epsilon_i\cdot q}{\bar z-\bar z_i}\right]\prod_{i=1}^{n}|z-z_i|^{\alpha' q\cdot k_i}$$

Expansion in q: All integrals involved related to $\mathcal{I}_i^j$ by IBP and PF identities.
Warning: Through $\mathcal{O}(q)$ appear 24 new types of kinematic structures.

…Once the dust has settled, the symmetric case yields [II]:

$$\hat{S}^{(\mu\nu)}(q,k_i) = \sum_{i=1}^{n}\left[\frac{k_i^{\mu}k_i^{\nu}}{q\cdot k_i} - i\frac{k_i^{\mu}q_\rho J_i^{\nu\rho}}{q\cdot k_i} - \frac{1}{2}\frac{q_\rho J_i^{\mu\rho}q_\sigma J_i^{\nu\sigma}}{q\cdot k_i} + [\hat\eta]_i^{\mu\nu} + [\hat{\alpha'}]_i^{\mu\nu}\right]$$

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Soft **graviton**: $\epsilon^{g}_{\mu\nu}\,\hat{S}^{\mu\nu}_{\text{bos.}} = \hat{S}^{\text{graviton}}_{\text{field theory}} + \epsilon^{g}_{\mu\nu}[\hat{\alpha'}]^{\mu\nu}$

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The Superstring Story and the Question of Universality

- Observations [III]

$$a) \quad \mathcal{M}_{n+1} = \mathcal{M}_n * S = \mathcal{M}_n * (S_{\text{bos.}} + S_{\text{susy}})\,,$$
$$b) \quad \mathcal{M}_n = \mathcal{M}_n^b * \mathcal{M}_n^s\,,$$

No new integrals appear in S_{susy} through $\mathcal{O}(q^2)$!

$$S_{\text{bos.}} = \epsilon^S_{\mu\nu} \sum_{i=1} \frac{k_i^\mu k_i^\nu}{k_i \cdot q} + \mathcal{O}(q^0)\,, \quad S_{\text{susy}} = 0 + \mathcal{O}(q^0)$$

$$S_{\text{bos.}}\big|_{\mathcal{O}(q^0)} \sim -i\epsilon^S_{\mu\nu} \sum \frac{k_i^\mu q_\rho}{q \cdot k_i} J_i^{\nu\rho} \mathcal{M}_n^b\,, \quad S_{\text{susy}}\big|_{\mathcal{O}(q^0)} \sim -i\epsilon^S_{\mu\nu} \sum \frac{k_i^\mu q_\rho}{q \cdot k_i} J_i^{\nu\rho} \mathcal{M}_n^s$$

$$S_{\text{bos.}}\big|_{\mathcal{O}(q)} \sim (\hat{S}^{(1)} + [\hat{\alpha'}])\mathcal{M}_n^b\,, \quad S_{\text{susy}}\big|_{\mathcal{O}(q)} \sim \hat{S}^{(1)} \mathcal{M}_n^s + \text{“}(J\mathcal{M}_n^b)(J\mathcal{M}_n^s)\text{”} - [\hat{\alpha'}]\mathcal{M}_n^b$$

- Soft theorem for a symmetric state in superstrings

$$\mathcal{M}_{n+1}^{(\mu\nu)} = \left(\hat{S}^{(\mu\nu)}_{\text{bos.}}\big|_{\alpha'=0} \right) \mathcal{M}_n + \mathcal{O}(q^2)$$

Equivalent to field theory! But $\neq$ Bosonic (and Heterotic) string.

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Equivalent to field theory! But $\neq$ Bosonic (and Heterotic) string.

The Superstring Story and the Question of Universality

- Observations [III]

$$a) \quad \mathcal{M}_{n+1} = \mathcal{M}_n * S = \mathcal{M}_n * (S_{\text{bos.}} + S_{\text{susy}}) \,,$$
$$b) \quad \mathcal{M}_n = \mathcal{M}_n^b * \mathcal{M}_n^s \,,$$

No new integrals appear in S_{susy} through $\mathcal{O}(q^2)$!

$$S_{\text{bos.}} = \epsilon^S_{\mu\nu} \sum_{i=1} \frac{k_i^\mu k_i^\nu}{k_i \cdot q} + \mathcal{O}(q^0) \,, \quad S_{\text{susy}} = 0 + \mathcal{O}(q^0)$$

$$S_{\text{bos.}}\big|_{\mathcal{O}(q^0)} \sim -i\epsilon^S_{\mu\nu} \sum \frac{k_i^\mu q_\rho}{q \cdot k_i} J_i^{\nu\rho} \mathcal{M}_n^b \,, \quad S_{\text{susy}}\big|_{\mathcal{O}(q^0)} \sim -i\epsilon^S_{\mu\nu} \sum \frac{k_i^\mu q_\rho}{q \cdot k_i} J_i^{\nu\rho} \mathcal{M}_n^s$$

$$S_{\text{bos.}}\big|_{\mathcal{O}(q)} \sim (\hat{S}^{(1)} + [\hat{\alpha'}])\mathcal{M}_n^b \,, \quad S_{\text{susy}}\big|_{\mathcal{O}(q)} \sim \hat{S}^{(1)} \mathcal{M}_n^s + \text{“}(J\mathcal{M}_n^b)(J\mathcal{M}_n^s)\text{”} - [\hat{\alpha'}]\mathcal{M}_n^b$$

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Three-point Amplitudes and Low-Energy Actions

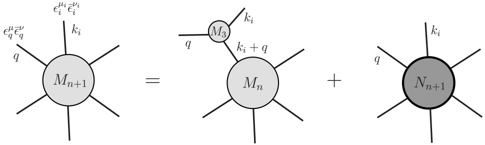
Bosonic, Heterotic and Type II Superstring low-energy actions ($\lambda_0 = \frac{1}{4}, \frac{1}{8}, 0$)
[Zwiebach, **PLB156**, '85], [Metsaev, Tseytlin, **NPB293**, '87]

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-G} \Big\{ R - G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} e^{-\frac{4}{\sqrt{D-2}}\phi} H_{\mu\nu\rho} H^{\mu\nu\rho}$$
$$+ \alpha' \lambda_0 \, e^{-\frac{2}{\sqrt{D-2}}\phi} \Big[R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 + \cdots \Big] + \mathcal{O}(\alpha'^2) \Big\}$$

Amplitude of three massless closed bosonic or super strings ($\lambda_0 = \frac{1}{4}, 0$)

$$\mathcal{M}_3^{\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3}(k_1,k_2,k_3) = 2\kappa_D \left(\eta^{\mu_1\mu_2} k_1^{\mu_3} + (\circlearrowright 1,2,3) + 2\alpha' \lambda_0 k_1^{\mu_3} k_2^{\mu_1} k_3^{\mu_2} \right) \times \left(\mu_i \leftrightarrow \nu_i \right)$$

For the heterotic case take (bosonic)×(supersymmetric).



On-shell gauge invariance fixes soft theorems of the graviton and dilaton!

[Bern, Davies, Di Vecchia, Nohle '14], [Di Vecchia, Marotta, Nohle, M. '15], [Di Vecchia, Marotta, M. '16]

The Dilaton Soft Theorem - Tree-level Universality!

- 1975: Renormalization of dual resonance models
  [Shapiro], [Ademollo, D'Adda, D'Auria, Gliozzi, Napolitano, Sciuto, Di Vecchia]
- **New** dilaton soft theorem through **ssLO**:

$$\epsilon_\phi^{\mu\nu}\, [\hat{\alpha'}]_{\mu\nu} \stackrel{!}{=} 0$$

- **Universally** and for $k_i^2 = -m_i^2$, we observe [II]

$$\epsilon_\phi^{\mu\nu} \hat{S}_{\mu\nu} \propto 2 - \sum_{i=1}^n \mathcal{D}_i + \frac{q_\rho}{2} \sum_{i=1}^n \mathcal{K}_i^\rho + \sum_{i=1}^n \frac{1}{2} \frac{q^\rho q_\sigma}{q \cdot k_i} \left[\mathcal{S}_{i,\rho\mu} \mathcal{S}_i^{\mu\sigma} + d \left(\epsilon_{i\rho} \frac{\partial}{\partial \epsilon_{i\sigma}} + \bar{\epsilon}_{i\rho} \frac{\partial}{\partial \bar{\epsilon}_{i\sigma}} \right) \right]$$

$$- \sum_{i=1}^n \frac{m_i^2}{q \cdot k_i} \left[1 + q^\rho \frac{\partial}{\partial k_i^\rho} + \frac{1}{2} q^\rho q^\sigma \frac{\partial^2}{\partial k_i^\rho \partial k_i^\sigma} \right]$$

Local operators are purely conformal transformations:

$$\mathcal{D}_i = k_i^\mu \frac{\partial}{\partial k_i^\mu} \,,$$

$$\mathcal{K}_i^\rho = k_i^\rho \frac{\partial^2}{\partial k_i^\mu \partial k_{i\mu}} - 2 k_i^\mu \frac{\partial^2}{\partial k_i^\mu \partial k_{i\rho}} + i \mathcal{S}_i^{\mu\rho} \frac{\partial}{\partial k_i^\mu}$$

Scalar pole-terms are mass-couplings:

$$-2 m_i^2 \frac{1}{(k_i + q)^2} M_n(k_i + q)$$

…empirical observation! Hidden symmetry?

$$S_{\text{NG dilaton}} = \sum_{i=1}^n \left[\frac{D - nd}{n} - \mathcal{D}_i + \frac{q_\mu}{2} \left(\mathcal{K}_i^\mu - 2d\, \partial_{k_i}^\mu \right) \right]$$

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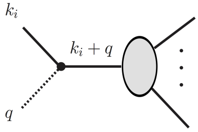

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New soft theorem for the antisymmetric case ($B_{\mu\nu}$) (unpublished)

- Recall that leading term $\frac{k_i^\mu k_i^\nu}{k_i \cdot q}$ is manifestly symmetric.
- Subleading terms: $\exists$ in general a holomorphic soft theorem: [I]

$$M_{n+1}^{\mu\nu}\Big|_{\mathcal{O}(q^0)} = -i\sum_{i=1}^{n}\left[\frac{q_\rho \bar{k}_i^\nu J^{\mu\rho}}{k_i \cdot q} + \frac{q_\rho k_i^\mu \bar{J}^{\nu\rho}}{\bar{k}_i \cdot q}\right] M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)\Big|_{k=\bar{k}}$$

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Gauge invariance, $\epsilon^B_{q\,\mu\nu} \to \epsilon^B_{q\,\mu\nu} + q_\mu \chi_\nu - q_\nu \chi_\mu$, implies

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Leading to a physical sL soft theorem:

$$M_{n+1}^{[\mu\nu]} = -i\sum_{i=1}^{n}\left[\frac{k_i^{[\nu} q_\rho}{q \cdot k_i}(S_i - \bar{S}_i)^{\mu]\rho} - \frac{1}{2}(S_i - \bar{S}_i)^{\mu\nu}\right] M_n(k_i, \epsilon_i, \bar{\epsilon}_i) + \mathcal{O}(q)$$

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Summary and Conclusions

- Studied soft behavior of $\mathcal{M}_{n+1}^{\mu\nu}$ in different string theories through $\mathcal{O}(q)$.
- Found generic ssL soft theorem for $\mathcal{M}_{n+1}^{(\mu\nu)}$

$$\mathcal{M}_{n+1}^{(\mu\nu)} = \sum_{i=1}^{n} \left[\frac{k_i^\mu k_i^\nu}{q \cdot k_i} - i \frac{k_i^\mu q_\rho J_i^{\nu\rho}}{q \cdot k_i} - \frac{1}{2} \frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{q \cdot k_i} + [\hat{\eta}]_i^{\mu\nu} + [\hat{\alpha'}]_i^{\mu\nu} \right] \mathcal{M}_n(k_i, \epsilon_i, \bar{\epsilon}_i) + \mathcal{O}(q^2)$$

- Higher-order effective operators modify graviton soft theorem at ssL graviton soft theorem is different in bosonic/heterotic/superstring!
- The dilaton soft theorem remains the same in all string theories! Surprising observation: Contains the space-time generators of conformal transformations!? (resembling the NG dilaton)
- Found a sL soft theorem for $\mathcal{M}_{n+1}^{[\mu\nu]}$; true (gauge) obstruction at $\mathcal{O}(q)$.

$$\mathcal{M}_{n+1}^{[\mu\nu]} = -i \sum_{i=1}^{n} \left[\frac{k_i^{[\nu} q_\rho}{q \cdot k_i} (S_i - \bar{S}_i)^{\mu]\rho} - \frac{1}{2} (S_i - \bar{S}_i)^{\mu\nu} \right] \mathcal{M}_n(k_i, \epsilon_i, \bar{\epsilon}_i) + \mathcal{O}(q)$$

- Last tree-level step: The RR sector of Type II superstrings? (in progress)
- Loops? Low-energy action of superstrings?